# An Indefinitely Extensible Model of Dependent Type Theory with One Extensible Universe

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**Abstract**

We present *extensible systems* as a model of dependent type theory, providing an interpretation of types in which both types and terms are considered entities that can grow indefinitely. Extensible systems are closely related to direct and inverse systems. References to an extensible system are context-dependent and may yield extensionally different but structurally indistinguishable states.
In this model, we carry out the *dependent function space*, one of the standard constructions of dependent type theory. Other type-forming operations, such as dependent sums or inductive types, as well as an explicit treatment of substitutions, are not covered here and are left to future work.
The model supports the interpretation of a universe type ***type*** and, in particular, the judgment ***type* : *type***. This is achieved by treating universe levels as part of the staging structure of an extensible system. Since typing judgments describe only the structural layer, ***type*** is interpreted as a single universe with extensionally different but structurally identical states.
The model allows the *potential infinite* to be formalized when all infinite classes are considered indefinitely extensible. This provides a basis for a mathematics in which infinity is viewed as a dynamic process rather than the cardinality of a fixed set.

**Keywords:** Indefinite extensibility, Potential infinite, Model theory, Dependent type theory, Universe, Proof assistant

## 1 Introduction

Indefinite extensibility is often described in a setting of modal logic; see, for example, [1–4]. Our approach, by contrast, is grounded in a more direct conception and

describes indefinite extensibility dynamically. We therefore take the dynamic language in mathematics more literally, so we try to make sense of the idea that there are mathematical objects and statements which do not exist now, but will exist in future. Another way to phrase this is that we appeal to an idealized mathematician situated in time, in the spirit of Heyting's idealized intuitionistic mathematician [5]. However, our approach is purely semantic; we do not make use of intuitionistic or semi-intuitionistic logic [6].

We develop a denotational semantics for dependent type theory with a single universe, without appeal to any term model and evaluation system. This model is based on classes, formalized as extensible systems (a notion that we define here), and allows for the interpretation of the judgment $type : type$, in which universe levels appear only as states of one extensible universe.[1] It is worth noting that extensibility may apply to all types, including the language itself, so that the type theory can be regarded as an open-ended framework [9].

## 1.1 Indefinite Extensionality in Set Theory

Set theory, founded on the first-order axioms of ZFC, is widely regarded as the foundation of mathematics. In this view, the mathematical universe constitutes a model of the ZFC axioms, interpreted according to Tarskian model theory. Yet, there exists an irreconcilable contradiction between the two rôles that set theory plays: as the foundational framework of mathematics itself and as a Tarskian model of the ZFC axioms. In its foundational rôle, set theory underlies model theory — every model's carrier set is assumed to lie within the set-theoretic universe $\mathcal{V}$. However, models of the ZFC axioms cannot be based on a single carrier set themselves, due to the indefinite extensibility of the concept of the "set of all sets".

The concept of a set cannot allow for indefinite extensibility, since a set is precisely defined by the totality of its elements. Adding a new element to a given set necessarily produces a new set; one cannot, in any meaningful sense, "extend" a set by the addition of further elements.

The iterative conception of the set-theoretic universe, given by the cumulative hierarchy $(\mathcal{V}_\alpha)_{\alpha \in On}$, provides an alternative. However, this conception conflicts with the idea of a single, completed universe, and thus cannot serve as a Tarskian model of the ZFC axioms. A model theory with extensible domains for first-order logic was introduced in [10], and the present paper builds on that approach.

## 1.2 Sets, Classes and Systems

We begin by introducing some notation consistent with standard set-theoretic formulations. A set denotes a specific, fixed, and extensionally definite collection of objects. By contrast, a class refers to a collection of objects that need not be fixed or extensionally definite, and is assumed to be the extension of a predicate. A proper class is

[1] It is well known that a naïve treatment of $type : type$ leads to inconsistency, as shown by Girard's paradox. The dynamic approach avoids this issue, as not every typeable term is interpreted. Variables of the form $\varrho \to \sigma$, where both types are indefinitely extensible (for instance $type \to type$), may yield terms that admit a typing judgment but are not interpretable in extensible systems. Such variables are however necessary for the formulation of the paradox, for example, in the definition of a *universal system of notations for relations* [7], or in the construction of a *powerful universe* [8].

a class that does not form a set. An important instance of a proper class (of objects of some type) is that of an *indefinitely extensible* class. Throughout this paper, we consider only classes of this kind.

A widely accepted view provides a starting point for understanding the relationship between sets, classes, and systems as it is applied here: the set-theoretic universe $\mathcal{V}$ is a proper class, usually presented as the cumulative hierarchy $(\mathcal{V}_\alpha)_{\alpha\in On}$. It appears intuitively clear — and is supported by the set-theoretic reflection principle — that one never requires all sets within this universe. One may restrict oneself to a sufficiently large part of $\mathcal{V}$, which then constitutes a set. Alternatively, $\mathcal{V}$ may be regarded as a transitive model (a set) of a larger universe with an extended class of ordinals $On$. In concrete contexts, one may also select a specific inaccessible cardinal $\alpha$ and take the set $\mathcal{V}_\alpha$ in place of $\mathcal{V}$.

From this description, we extract the idea that a proper, indefinitely extensible class $\mathcal{M}$ is given as a system $(\mathcal{M}_i)_{i\in\mathcal{I}}$ — systems being described in Section 1.3, and more formally in Section 2.3. The main characteristic of an extensible class is that, in any given context, its extension can be fixed so as to yield a sufficiently large set (relative to that context). This set is structurally identical to the class in the sense that no structural distinctions can be detected. Such a set may be specified either implicitly, or explicitly by reference to a sufficiently large stage $i \in \mathcal{I}$ (of which there are several). This situation is captured by the following reflection principle:

> Every indefinitely extensible class possesses states as sets that are structurally indistinguishable from the class itself.

We *use* this reflection principle, which simultaneously constitutes the *main theorem* of the theory. A version of this principle for simple type theory was formulated in [11]; the corresponding formulation for dependent type theory will be presented in a subsequent paper.

Unlike a set, a class is not itself a mathematical entity. When we refer to a proper class, formalized as a system $\mathcal{M}_\mathcal{I}$, we can only refer to specific states $\mathcal{M}_i$, never to $\mathcal{M}_\mathcal{I}$ as a single, completed object, nor to a totality of (absolutely) all elements in all sets $\mathcal{M}_i$. Thus, a class is a meta-level concept which, from a syntactical perspective, coincides with a predicate. It coincides with that of a system when its dynamic behavior is formalized. And in order to speak of a class as a single entity, we introduce the notion of a limit of a system. The essential difference from the standard notion of a limit is that, in our approach, limits are context-dependent rather than absolute. Indefinite extensibility ensures that further states beyond any given limit may exist.

Our contextual conception of the relation between sets and classes differs from that of standard set theory, in which a class lacks a fixed extension merely because it is considered "too large". This line of reasoning, as noted by Shapiro and Wright [12], is not convincing.[2] Our conception is somewhat closer to Zermelo's perspective [14] (though it differs in important respects), according to which the distinction between a set and a proper class depends on the stage from which it is viewed: what appears as a proper class at one stage may become a set at a higher inaccessible rank.

[2] Moreover, ZFC set theory conflates distinct conceptual notions within a single axiomatic framework for sets. Peter Fletcher [13] has examined this critically and concludes that ZFC is a form of *axiom-hacking*.

## 1.3 Systems and their Limits

The dynamic aspect of an indefinitely extensible class is formalized as a system, that is, a family $\mathcal{M}_{\mathcal{I}} := (\mathcal{M}_i)_{i\in\mathcal{I}}$ of sets $\mathcal{M}_i$, equipped with a predecessor relation $\overset{p}{\mapsto}$. A system is thus a pair $(\mathcal{M}_{\mathcal{I}}, \overset{p}{\mapsto})$, where $(\mathcal{I}, \leq)$ is a directed class whose elements are the *stages* or *indices*.[3] For each stage $i \in \mathcal{I}$, the set $\mathcal{M}_i$ represents the state of the system (or class) at that stage. The predecessor relation $\overset{p}{\mapsto}$ can be understood as a non-symmetric equality relation connecting the states $a_i$ of a changing element $a$. This formalization does not presuppose that $a$ exists as a single, unified entity; rather, it is conceived solely as a bundle of states.

As discussed in Section 1.2, there are no entities called classes; they exist only informally. When we refer to a class $\mathcal{M}$, we either make the stages explicit, referring to it via a set $\mathcal{M}_i$, or we appeal to a sufficiently large set $\mathcal{M}$, where the relevant stage is implicit. Whenever we refer implicitly, on the meta-level, to a sufficiently large set $\mathcal{M}$, it is understood to depend on the current context of investigation. Within any mathematical investigation, we may also make explicit reference to particular states $\mathcal{M}_i$, and there may be finitely many such references. Since the index class $(\mathcal{I}, \leq)$ is directed, we can assume that the implicitly chosen index corresponding to $\mathcal{M}$ is an upper bound of all the indices that are explicitly mentioned.

The formalization of an indefinitely large state is provided by the limit of the system. The system $\mathcal{M}_{\mathcal{I}}$ captures the dynamic aspect, while its limit represents the indefinitely large set — one that lies beyond the system, though not in an absolute sense. The construction of a limit always occurs within some context: either at the meta-level, relative to the current investigation, or at object-level within a formal context. For simple type theory the latter have been presented in [11].

A limit $\mathcal{M}$ ensures that as an investigation progresses and additional explicit indices are introduced, possibly necessitating a larger set $\mathcal{M}$, the implicitly determined index of $\mathcal{M}$ can always be taken to be larger than all explicitly given indices. This is a basic property of a limit set. Expressed more informally and dynamically, whenever new indices are introduced explicitly, the sufficiently large index is silently enlarged in the background of the investigation, as needed. This is possible because both the construction and reference to a limit are context-dependent. Consequently, two occurrences of a sufficiently large (i.e., limit) set need not have the same extension, but they do share the same structure. This is particularly relevant in the case of a self-contained universe; see Example 3.41.

Limits are also essential for identifying a class or system as a single entity. This fact is taken into account when considering classes of classes, and in particular when an entire system is used within an index (see Section 1.5). Another fundamental property of limits is that they guarantee *consistency* (or, equivalently, *coherence*) among interpretations at different stages. There are two important aspects:

[3] The notion of a system presupposes that its index class $\mathcal{I}$ is intensionally definite, as with all of our notions. If a system that describes a proper class is only given by its successive states — say $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots$ — it is not clear a priori whether the corresponding index class is $\mathbb{N}$ or a superclass thereof, such as $\mathbb{Q}$, $\mathbb{R}$, or $On$. This kind of indefiniteness prevents, for example, the definition of an indefinitely-many filter (Definition 2.13): when only initial segments of an increasing class are available, the notion of a cofinal subclass becomes unclear.

1. When referring to an object $a$ at stages $i$ and $j$ by its respective states $a_i$ and $a_j$, we assume these are approximations of the same underlying object $a$. This assumption is reflected in the requirement that the stages form a directed class, and that $a$ exists at indefinitely many stages (in particular, at cofinal many stages). Therefore, there is a stage $i'$ such that $i, j \leq i'$, and a corresponding approximation $a_{i'}$ of $a$ at stage $i'$. Formally, this means that $a_{i'} \overset{p}{\mapsto} a_i$ and $a_{i'} \overset{p}{\mapsto} a_j$, which we denote by $a_i \asymp a_j$. If this is case, then $a_i$ and $a_j$ are consistent states because they share a common future state.
2. A statement about objects is represented by a term $\Phi$ of type *bool* (or, more generally, by a type $\Omega$ for truth values) in the type theory, corresponding to a formula. Such formulas are treated like any other object within the theory. It is possible for a formula to hold when an object is considered at stage $i$, but not at stage $j$. A limit stage $i' \geq i, j$ guarantees that the truth value of $\Phi$ at $i'$ is preserved in subsequent extensions.

The construction of a limit involves two steps. First, the elements of the limit are defined. Then, the limit itself is specified as the set of all those elements. Both steps involve constructing *sets*. The elements of a limit are formed as maximal consistent sets $\alpha$ of states. Such a collection of states, however, need not be extensionally definite at first. Consequently, distinct limit elements as well as the limit sets usually differ extensionally, depending on the context in which they are constructed. Nevertheless, the universal property of a limit guarantees that these sets are structurally identical.

Consider the limit $\mathcal{M} = \{\alpha \mid \alpha \text{ a maximal consistent set of states}\}$ of a system $\mathcal{M}_{\mathcal{I}}$. Suppose we adjoin this limit on top of the system — details will be presented in Section 2.4. Adding this limit preserves the basic properties of an extensible system. After adding the limit, we can then extend the system beyond this limit without fixing a specific extension, which makes the system open and extensible again. Thus, the extension of a system involves two complementary steps:

1. Compactification of the system, whereby it becomes temporarily definite, i.e., makes it a set.
2. Reconstitution of the system as an indefinitely extensible class.

These two steps result in two *modes* of the system, and it can be assumed that a system extends by alternating between these modes. The extension of a system occurs in parallel with the extension of its index class. Accordingly, we obtain an increasing sequence of stage classes, in most cases as simple subset inclusions $\mathcal{I} \subseteq \bar{\mathcal{I}} \subseteq \mathcal{I}'$.[4] We may describe the extension process as follows:

$$
\begin{array}{lll}
 & \mathcal{M}_{\mathcal{I}} = (\mathcal{M}_i)_{i \in \mathcal{I}} & \text{an indefinite extensible class (i.e., extensible system),} \\
\rightsquigarrow & \mathcal{M} = \mathcal{M}_{top} & \text{a set representing } \mathcal{M}_{\mathcal{I}} \text{ (i.e., a limit),} \\
\rightsquigarrow & (\mathcal{M}_i)_{i \in \bar{\mathcal{I}}} & \text{with } \bar{\mathcal{I}} = \mathcal{I} \cup \{top\}, \text{ the compactification,}
\end{array}
$$

[4] To illustrate the contrast with the usual perspective, consider the index class $\mathcal{I}$ of stages representing time. In the usual view, the set of points in time — past, present, and future — is fixed. The flow of time is then imagined as a point in time moving towards the future while the overall temporal structure remains unchanged. In the view adopted here, however, the future does not exist at all. The flow of time not only adds new historical states, but also extends the class of points in time itself.

$$\rightsquigarrow \quad \mathcal{M}_{\mathcal{I}'} = (\mathcal{M}_i)_{i\in\mathcal{I}'} \qquad \text{with } \bar{\mathcal{I}} \subseteq \mathcal{I}', \text{ again an indefinite extensible class.}$$

A formal representation of an indefinitely extensible class consists of a system, $\mathcal{M}_{\mathcal{I}}$, together with its limit, $\mathcal{M}$. When we refer to a class as a set, we are referring to this limit, which we also call a sufficiently large set. Due to the contextual nature of limits, a class should not be seen as formalized by a single system and its limit, but rather by all possible systems and their limits.

## 1.4 The Potential Infinite

In type theory, the universe type *type* is the primary example of an indefinitely extensible type. Types such as *nat* (the type of natural numbers) are generally considered definite. However, we make no assumptions about the definiteness of types in general, except that types of finite sets — such as $\mathbb{B} = \{true, false\}$ — are definite. Accordingly, treating all infinite classes as indefinitely extensible yields a formalisation of the potential infinite:

$$\text{Potential infinite} \;=\; \text{Indefinite extensibility} \;+\; \text{Finitistic view.}$$

Regarding only *type* as indefinitely extensible while treating all other base types as definite can be seen as one extreme on a spectrum of positions — one that avoids indefinite extensibility whenever possible. On the other end of the spectrum is the potentialist perspective. The model presented here accommodates both viewpoints: it allows certain types to be treated as extensible and enriched by stages and states. From a potentialist standpoint, the type *nat* is potential infinite, that is, indefinitely extensible, whereas from an infinitary standpoint, it is definite. Throughout the paper, we alternate between these perspectives, as the finitistic view provides intuitive examples.

*Example 1.1* Consider the potential infinite class of natural numbers, $\mathbb{N} = \{0, 1, \dots\}$. As a system, it can be represented as $(\mathbb{N}_i)_{i\in\mathbb{N}^+}$ with finite sets $\mathbb{N}_i = \{0, 1, \dots, i-1\}$ and index class $\mathbb{N}^+ = \{1, 2, \dots\}$.[5] The limit of the system, $\mathbb{N}$, is then interpreted as a sufficiently large finite set.

Describing the index class as a system immediately would lead to an infinite regress. This is avoided by distinguishing between the object-level and the meta-level. The objects under investigation reside within $\mathcal{M}_i$ — in the above example, $\mathbb{N}_i$ — while the index class $\mathcal{I}$ (here, $\mathbb{N}^+$) is treated as a meta-level construct. We therefore assume that $\mathcal{I}$ is an indefinitely extensible class, allowing reference to a sufficiently large set. So, from the very beginning, $\mathcal{I}$ is some set $\mathcal{I}_j$ for an implicitly given index $j$.[6]

At the object-level, however, indices must be mentioned explicitly, as this is the level at which the concepts are developed. In other words, indefinite extensibility is

[5] We use $\mathbb{N}^+$ instead of $\mathbb{N}$ as index class, because the index 0 would result in an empty set $\mathbb{N}_0$, which is not possible for a factor system when combined with nonempty sets, see Section 2.3.

[6] Alternatively, one may assume that classes are mathematical entities and work with them on the meta-level to develop (on object-level) a model theory based on systems and their limits. It can then be verified that all properties required of classes are equally satisfied by the limits of systems, so that a class can be consistently replaced by a sufficiently large set.

applied at the meta-level along with the associated notions of sets, classes, systems, stages and the reflection principle. The task is to make these ideas explicit at the object-level.

## 1.5 Families of Classes and Classes of Classes

In [15] we introduced the function space of systems; here, we extend this approach to dependent function spaces and universes. To this end, we need to describe families of classes and classes of classes as systems. Suppose we have a family $(\mathcal{N}^A)_{A\in\mathcal{M}}$ of classes, indexed over a class $\mathcal{M}$. To make this family explicit as a system, any reference to an object in $\mathcal{M}$ must be given by a state $a_i \in \mathcal{M}_i$. For each such $a_i$, there is a corresponding fiber system $(\mathcal{N}_j^{a_i})_{j\in\mathcal{J}^{a_i}}$, formalizing the dynamic behavior of $\mathcal{N}^A$.

A reference to an element of this fiber also must be explicit: first by choosing a stage $j \in \mathcal{J}^{a_i}$, and then an element $b_j \in \mathcal{N}_j^{a_i}$. This gives rise to a function $\varphi$ that selects, for each $a_i \in \mathcal{M}_i$, a stage $j \in \mathcal{J}^{a_i}$ — that is, a dependent function $\varphi \in \Pi_{a_i\in\mathcal{M}_i}\,\mathcal{J}^{a_i}$. An index of the entire family is then a pair $(i,\varphi) \in \Sigma_{i\in\mathcal{I}}, \Pi_{a_i\in\mathcal{M}_i}, \mathcal{J}^{a_i}$. This construction will be refined further in Section 3.1.

Consider now a class $\mathcal{M}$ of classes, for instance a universe, that we wish to describe as a system of systems. Let $\mathcal{N}_\mathcal{J}$ be a system with limit $\mathcal{N}$ that formalizes a class in $\mathcal{M}$. How should we define the class $\mathcal{I}$ of stages when $\mathcal{M}$ is formalized as a system $\mathcal{M}_\mathcal{I}$? The basic idea here is that $\mathcal{M}$ can grow in two ways: first, by the addition of new classes to $\mathcal{M}$, and second, by the growth of an existing system $\mathcal{N}_\mathcal{J}$. This already requires that we can identify a growing system within $\mathcal{M}$ and assign a stage to it. A system as a whole can be identified via its limit[7], but a limit itself does not possess a stage. Any reference to a stage must therefore refer to the stage of a set within the system.

Let us first assume that no new elements can be added to $\mathcal{M}$, but that the elements within $\mathcal{M}$ may grow. In this case, the class $\mathcal{M}$ is implicitly represented by a limit set $\mathcal{M}$, which contains the limit sets $\mathcal{N}$ of its elements. An explicit assignment of a state $\mathcal{M}_i$ of the class $\mathcal{M}$ should then be the totality of all states of its constituent elements $\mathcal{N}$. This can be formalized by a function $\varphi \in \Pi_{\mathcal{N}\in\mathcal{M}}\,\mathcal{N}^{ix}$, where $\mathcal{N}^{ix}$ denotes the index class $\mathcal{J}$ of the system $\mathcal{N}_\mathcal{J}$ with limit $\mathcal{N}$.

Now assume that $\mathcal{M}$ may also grow indefinitely. In this case, a stage of $\mathcal{M}$ must first specify the set of contained sets. To this end, we select a subset $V \subseteq \{\mathcal{N}, \mathcal{N}', \dots\}$ of the possible limit sets in $\mathcal{M}$, and we refer to the class of all such fixed sets $V$ as $\mathcal{P}_{set}(\mathcal{M})$. Relative to this set $V$, we define the stages of each $\mathcal{N} \in V$, as described above.

This gives the index class $\Sigma_{V\in\mathcal{P}_{set}(\mathcal{M})}\,\Pi_{\mathcal{N}\in V}\,\mathcal{N}^{ix}$ for $\mathcal{M}$. A minor modification is required to handle the empty set in $\mathcal{P}_{set}(\mathcal{M})$; this will be discussed in detail in Section 3.6. The state of $\mathcal{M}$ at $(V,\varphi)$ is therefore $\mathcal{M}_{(V,\varphi)} = \{\mathcal{N}_{\varphi(\mathcal{N})} \mid \mathcal{N} \in V\}$. For systems of systems, we thus obtain the same kind of index as for a family of systems, and indeed, a system of systems can also be described as a family of systems.

[7] As an alternative to using limits to identify a system, one may assign a name to each system. The interpretation of a type would then be a pair consisting of the (normal form or an equivalence of the) type and its value. This involves gluing models [16], which we will not consider here.

# 2 Preliminaries

Before developing the extensible model of dependent type theory, we briefly recall the basic notions of its syntax and outline the notion of a categorical model. We then introduce the concept of an extensible system, along with further concepts necessary for dependent types and universes. In doing so, we revisit the notion of a factor system, already developed for simple type theory.

## 2.1 The Type Theory

We do not fix a particular type theory here; rather, we consider a dependent type theory that admits extensions via a (typed) signature [17–19]. Let $\Sigma$ denote a signature, including the type formers. The type theory is assumed to be strong enough to include the usual type formers, in particular $\Sigma$-types and $\Pi$-types. We further assume that the declarations $\mathit{unit} : \mathit{type}$, $\mathit{void} : \mathit{type}$, $\mathit{bool} : \mathit{type}$ are included in the signature, and unless stated otherwise, it starts with $\mathit{type} : \mathit{type}$. In examples we often make use of other base types such as $\mathit{nat} : \mathit{type}$ and $\mathit{On} : \mathit{type}$, the type of natural numbers and ordinal numbers, respectively. In the sequel, formulas are treated as terms of type $\mathit{bool}$.[8]

Let a fixed list of variables $\mathsf{x}_0, \mathsf{x}_1, \ldots$ be given according to the de Bruijn level convention. Typing contexts $\Gamma$ are then lists of types $\langle \varrho_0, \ldots, \varrho_{n-1} \rangle$, which may also be written explicitly as $\langle \mathsf{x}_0 : \varrho_0, \ldots, \mathsf{x}_{n-1} : \varrho_{n-1} \rangle$. For readability, we make use of alternative variable names and standard conventions, including infix notation and uncurrying. We denote the empty context by $\langle\rangle$, context extension by $\Gamma.\varrho$, and concatenation of two contexts by $\Gamma\Delta$. The basic judgment is the typing judgment

$$\Sigma; \Gamma \vdash \mathsf{r} : \varrho,$$

which asserts that, within a given signature $\Sigma$ and context $\Gamma$, the term $\mathsf{r}$ has type $\varrho$. In most cases, we assume the signature $\Sigma$ to be fixed and therefore omit it from the judgment, writing simply $\Gamma \vdash \mathsf{r} : \varrho$. We write

$$\Gamma \vdash \varrho$$

to mean that $\varrho$ is a type in context $\Gamma$. Formally, this corresponds to the judgment $\Gamma \vdash \varrho : \mathit{type}$, with $\varrho$ possibly subject to restrictions by some grammar. Although not

[8] One may also allow a more general type $\Omega$ of truth values in place of *bool*. If $\Omega$ is a definite type, there is no semantic difference: exactly the same formulas are interpretable, and the appropriate interpretation of the connectives and quantifiers depends on the inference rules one wishes to be sound in the model. The situation changes if $\Omega$ is itself indefinitely extensible, as is standard in intuitionistic logic. In that case, variables of type $\varrho \to \Omega$ are interpretable only when $\varrho$ is a definite type (see footnote 1). Under the additional finitistic stance described in Section 1.4 — where types such as *nat* are treated as indefinitely extensible — the impact is significant: variables of type $\mathit{nat} \to \Omega$, which are used for quantification over sets of natural numbers, are no longer interpretable.
The choice of *bool* also does not presuppose a realist philosophical position. Indeed, the model is philosophically neutral; it does not commit to any particular stance on the nature of mathematical objects. If a position were to be assigned, it would take the creative idealized mathematician (cf. Section 1) into account: mathematical objects, knowledge about them, and even the index structure formalizing time come into existence through an interdependent process of construction, unfolding in a future that does not yet exist, rather than being treated as fixed and observer-independent. In this sense, the model has a constructive character, even when *bool* is taken as the type of truth values.

directly needed here, we assume that standard properties such as subject reduction, strong normalization, and canonicity hold [20].

The logical connectives and quantifiers can be defined in the usual way.[9] Membership $\mathsf{r} \in \mathsf{s}$ is defined as $\mathsf{sr}$ whenever $\Gamma \vdash \mathsf{r} : \varrho$ and $\Gamma \vdash \mathsf{s} : \varrho \to \mathit{bool}$, and other set- (or class-)theoretic notations, such as $\mathsf{r} \subseteq \mathsf{s}$, are defined analogously. These are merely notational conventions; from a purely syntactical perspective, a class is simply a predicate. We do not introduce additional judgments for equality; the reader may read $=$ as Leibniz equality. We often use a subset type

$$\Gamma \vdash \varrho_{\,\Phi} := \{\mathsf{x} : \varrho \mid \Phi(\mathsf{x})\}$$

for a (probably dependent) predicate $\Gamma \vdash \Phi : \varrho \to \mathit{bool}$.

*Example 2.1* We present some types and terms, along with their interpretations, which will be used in subsequent examples.

| | |
|---|---|
| $\mathit{nat}^+ := \{\mathsf{x} : \mathit{nat} \mid \mathsf{x} > 0\}$ | $\mathbb{N}^+ := \{1, 2, \dots\}$, |
| $\langle \mathsf{i} : \mathit{nat}^+ \rangle \vdash \mathit{nat}_{\mathsf{i}}$ | $\mathbb{N}_i := \{0, \dots, i-1\}$, |
| $\mathit{seq} := \mathit{nat} \to \mathit{nat}_2$ | $\mathit{Seq}$ (binary sequences), |
| $\langle \mathsf{i} : \mathit{nat} \rangle \vdash \mathit{seq}_{\mathsf{i}} := \mathit{nat}_{\mathsf{i}} \to \mathit{nat}_2$ | $\mathit{Seq}_i$ (binary sequences of length $i$). |

Let constants

$$\mathsf{isSet}_\varrho : (\varrho \to \mathit{bool}) \to \mathit{bool}$$

be given, where $\mathsf{isSet}_\varrho(\Phi)$ asserts that a subclass $\Phi$ of elements of type $\varrho$ forms a fixed totality of objects. This is not necessarily a set, since all elements of a set must themselves be fixed (a set cannot contain a proper class, for otherwise its extension is not definite). The formula $\mathsf{isSet}(\Phi)$ asserts only that $\Phi$ forms a set, *assuming* that its elements are fixed; that is, it merely prohibits the addition of new elements. The notion $\mathsf{isSet}(\varrho)$ shall be a shorthand for $\mathsf{isSet}_\varrho(\lambda \mathsf{x}^\varrho\, \mathit{true})$.

Let the following predicate hold for all sequences that coincide with a given sequence at a specified position, and similarly for sequences of length $i$:

$$\langle \mathsf{x} : \mathit{seq} \rangle \vdash \mathsf{match} := \lambda \mathsf{y}^{\mathit{seq}}\, \exists\, \mathsf{k}^{\mathit{nat}}\, \mathsf{x}(\mathsf{k}) = \mathsf{y}(\mathsf{k}) : \mathit{seq} \to \mathit{bool},$$
$$\langle \mathsf{i} : \mathit{nat}^+, \mathsf{x} : \mathit{seq}_{\mathsf{i}} \rangle \vdash \mathsf{match}_{\mathsf{i}} := \lambda \mathsf{y}^{\mathit{seq}_{\mathsf{i}}}\, \exists\, \mathsf{k}^{\mathit{nat}_{\mathsf{i}}}\, \mathsf{x}(\mathsf{k}) = \mathsf{y}(\mathsf{k}) : \mathit{seq}_{\mathsf{i}} \to \mathit{bool}.$$

For a given type $\varrho$ define

| | |
|---|---|
| $\mathsf{nonEmpty}_\varrho := \exists \mathsf{x}^\varrho\, \mathit{true}$ | $: \mathit{bool}$, |
| $\mathit{sets}_\varrho := \lambda \mathsf{x}^\varrho\, \{\mathsf{y} : \varrho \to \mathit{bool} \mid \mathsf{isSet}_\varrho(\mathsf{y}) \wedge \mathsf{x} \in \mathsf{y}\}$ | $: \varrho \to \mathit{type}$. |

The predicate $\mathsf{nonEmpty}_\varrho$ returns $\mathit{true}$ if and only if the type $\varrho$ is inhabited. Every object of type $\mathit{sets}^{\mathsf{d}}_\varrho := \mathit{sets}_\varrho(\mathsf{d})$ contains $\mathsf{d}$ as an element. Its interpretation is

$$\mathcal{P}^d_{\mathit{set}}([\![\varrho]\!]) := \{N \in \mathcal{P}_{\mathit{set}}([\![\varrho]\!]) \mid d \in N\},$$

[9] In a follow-up paper, where we develop the type theory and its interpretation in more detail, we will treat the universal quantifier in a way that deviates from the standard interpretation in type theory. The reason is that the semantics of the universal quantifier cannot, in the setting of indefinite extensibility, be given by a continuous functional. Instead, its interpretation requires the use of the $\ll$-relation introduced in [10]. This point is essential for obtaining a sound interpretation, but it plays no role in the model theory developed in the present paper.

where $d := [\![\mathsf{d}]\!]$, and $\mathcal{P}_{set}(\mathcal{M})$ is as defined in Section 1.5. Categories provide another example constructed from types

$$obj \qquad \text{and} \qquad \langle \mathsf{x}_0 : obj, \mathsf{x}_1 : obj \rangle \vdash hom(\mathsf{x}_0, \mathsf{x}_1).$$

Then $\forall \mathsf{x}_0^{obj}\, \forall \mathsf{x}_1^{obj}\, \mathsf{isSet}(hom(\mathsf{x}_0, \mathsf{x}_1))$ asserts that the category is locally small, and if in addition $\mathsf{isSet}(obj)$ holds, the category is small. The term *small* is misleading in the context of this paper; a more accurate notion would be *fixed*.

In the definitions of terms such as $\mathsf{isSet}_\varrho$ and $\mathsf{nonEmpty}_\varrho$, the type $\varrho$ was treated as a meta-level variable. A dependent type theory equipped with a universe type, however, allows one to internalize $\varrho$ as an object-level variable. For example, the entire judgment scheme $\Gamma \vdash \varrho : type$ can then be expressed as a single instance of the variable rule $\langle \mathsf{x} : type \rangle \Gamma \vdash \mathsf{x} : type$. Strictly speaking, in this judgment the context is not $\Gamma$, but rather $\Gamma$ with an object-level variable $\mathsf{x}$ instead of the meta-level variable $\varrho$.

*Example 2.2* We may regard $\mathsf{nonEmpty}_\varrho$ as a dependent term, which allows us to use lambda abstraction and to introduce a dependent constant:

$$\mathsf{nonEmpty} := \lambda \mathsf{x}^{type}\, \exists \mathsf{y}^{\mathsf{x}}\, true : type \to bool,$$
$$\langle \mathsf{x} : type_{\,\mathsf{nonEmpty}} \rangle \vdash \mathsf{choose}(\mathsf{x}) : \mathsf{x}.$$

As in the meta-variable version, the term $\mathsf{nonEmpty}(\varrho)$ is equal to $true$ if and only if the type $\varrho$ is inhabited, in which case $\mathsf{choose}(\varrho)$ selects an element of $\varrho$. Specific instances may be defined explicitly, for example, $\mathsf{choose}(nat) = 0$ or $\mathsf{choose}(type) = unit$, so a term $\mathsf{choose}(\mathsf{x})$ is unnecessary for these particular cases. For the general treatment, however, as in $sets^{\mathsf{d}}_\varrho$, one can take $\mathsf{d} = \mathsf{choose}(\varrho)$ for a nonempty type $\varrho$.

## 2.2 The Model of Class-indexed Families of Classes

Most often, models of dependent types are presented in categorical terms. An early but accessible account to categorical models can be found in [21]. There are various notions, often equivalent, for instance category of families (abbreviated as CwF, [22, 23]) or natural models [24]. Nevertheless, the concrete categorical formulation is not relevant here since we are concerned with a specific model, which is basically Cartmell's original model of dependent type theory [16, 19, 25]. Cartmell's model is formulated using families of sets, whereas we work with families of classes. However, throughout this section we do not distinguish between classes and sets, since we focus solely on their structural behavior and do not take any "size issues" into account.

In most, if not all, categorical approaches one begins with a base category $\mathcal{C}$ of semantic contexts (denoted here as $X, Y, \dots$) and whose morphisms are semantic substitutions (denoted $\gamma, \delta, \dots$). On top of this base category, one defines semantic types and semantic terms, where we often omit the qualification "semantic":

| | |
|---|---|
| $Ty_X$ | types in context $X$, |
| $Tm_X(A)$ with $A \in Ty_X$ | terms of type $A$ in context $X$. |

Moreover, if $\mathcal{C}$ has a terminal object $\mathbf{1}$ and if the category is *contextual*, then each context $X$ corresponds to an iterated $\Sigma$-type in $\mathcal{U} := Ty_{\mathbf{1}}$. In particular, for the CwF

model, the semantic types and terms can be organized as a contravariant functor $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{F}am$, where $\mathcal{F}am$ denotes the category of families of classes.

**Definition 2.3** The category $\mathcal{F}am$ of class-indexed families of classes has pairs $(B, F_B)$ as objects, where $B$ is a class and $F_B = (F_b)_{b \in B}$ is a family of classes indexed over $B$. A morphism from $(B, F_B)$ to $(C, G_C)$ is a pair $(f, f_B)$ where $f : B \to C$ is a re-indexing function and $f_B = (f_b)_{b \in B}$ is a family of functions $f_b : F_b \to G_{f(b)}$.

The interpretation of types and terms correspond to the object part of $\mathcal{F}$:

$$\mathcal{F}(X) = (Ty_X, (Tm_X(A))_{A \in Ty_X}).$$

The presence of a universe type has the following effect. There are objects $U_X = [\![\Gamma \vdash \mathit{type}]\!] \in Ty_X$, where $X = [\![\Gamma]\!]$. Beside the types in $Ty_X$, there are also the terms of type *type*, interpreted as the elements in $Tm_X(U_X)$. In a Tarski-style universe each element $a \in Tm_X(U_X)$ serves as a code for a type $El(a) \in Ty_X$. In a Russell-style universe, which we consider here, no distinction is made between $a$ and $El(a)$.

To avoid inconsistencies, one usually assumes that there is an isomorphism between $Tm_X(U_X)$ and $Ty_X^{small} \subsetneq Ty_X$, the set of small types. So, in both approaches, there may be elements in $Ty_X$ that are not represented in $Tm_X(U_X)$ (or are not of the form $El(a)$). Most importantly, this includes the universe type itself. Consequently, the judgment $\Gamma \vdash \mathit{type} : \mathit{type}$ cannot hold, and one employs a hierarchy of universes instead.[10]

Seen from this categorical perspective, our aim is the identification $Tm_X(U_X) = Ty_X$, based on the idea that classes are indefinitely extensible. The following sections therefore develop the refinement of a class into a system. We begin, in the next section, with single classes and systems, and continue in Section 3 with families of classes and systems. In contrast to Definition 2.3, our focus here lies solely on the object part of the categorical structure; morphisms will play no role here.

## 2.3 Factor Systems

Indefinitely extensible classes are formalized via the notion of extensible systems, as introduced in Definition 2.14. These extensible systems are based on a factor system [11], which we present in this section.[11] It has been established that factor systems constitute models of the simply typed $\lambda$-calculus, as they permit the construction of function spaces. The function space between two factor systems $\mathcal{M}_{\mathcal{I}} := (\mathcal{M}_i)_{i \in \mathcal{I}}$ and $\mathcal{N}_{\mathcal{J}} := (\mathcal{N}_j)_{j \in \mathcal{J}}$ is given by (subsets of) the function spaces $[\mathcal{M}_i \to \mathcal{N}_j]$, indexed over pairs $(i, j) \in \mathcal{I} \times \mathcal{J}$. The treatment of dependent types requires a generalization of this construction to fibered function spaces, which forms the central topic of this paper.

[10] For example, in [26], a CwF hierarchy $Ty_0, Ty_1, \ldots$ of semantic types (and similarly of semantic terms) is used, together with an isomorphism $El_j : (Tm_X(U_j))_{j+1} \simeq (Ty_X)_j$ for a global section $U_j$ of $Ty_{j+1}$.

[11] Our presentation differs slightly from that of [11]. In particular, we refine the terminology. We do this by referring here to the index set as an index class. Moreover, we introduce a relation $a_i \equiv c_i$, which — though redundant in the presence of Property (Factor) — is more intuitive than the equivalent formulation $a_i \asymp c_i \wedge a_i, c_i \in \mathcal{M}_i$. The point is that consistency (denoted $\asymp$) relates elements at different stages, whereas $\equiv$ only applies to a single stage.

A *directed class* $(\mathcal{I}, \leq_\mathcal{I})$ is a class $\mathcal{I}$ equipped with a reflexive, transitive, and directed relation $\leq_\mathcal{I}$. We often write $\leq$ for $\leq_\mathcal{I}$ and simply $\mathcal{I}$ for $(\mathcal{I}, \leq)$. An *up-set* (or *up-class*) is a nonempty subset (or subclass, respectively) of $\mathcal{I}$ that is upward closed with respect to $\leq$. Moreover, $\uparrow i$ abbreviates $\{i' \in \mathcal{I} \mid i' \geq i\}$.

**Definition 2.4** A *system* $((\mathcal{M}_i)_{i\in\mathcal{I}}, \overset{p}{\mapsto})$, or simply $\mathcal{M}_\mathcal{I} := (\mathcal{M}_i)_{i\in\mathcal{I}}$, consists of sets $\mathcal{M}_i$, indexed over a nonempty directed class $\mathcal{I}$. The relation $a_{i'} \overset{p}{\mapsto} a_i$ is defined between elements $a_{i'} \in \mathcal{M}_{i'}$ and $a_i \in \mathcal{M}_i$ whenever $i \leq i'$, and it is reflexive, that is, $a_i \overset{p}{\mapsto} a_i$ holds for all $a_i \in \mathcal{M}_i$ and $i \in \mathcal{I}$. Throughout this paper, we adopt the convention that the index $i \in \mathcal{I}$ occurring in $a_i$ indicates that $a_i$ belongs to $\mathcal{M}_i$.

Let $a_i \equiv c_i$ be defined as $a_i \overset{p}{\mapsto} c_i$ and $c_i \overset{p}{\mapsto} a_i$ for $a_i, c_i \in \mathcal{M}_i$. Two elements $a_i \in \mathcal{M}_i$ and $a_{i'} \in \mathcal{M}_{i'}$ are said to be *consistent*, written $a_i \asymp a_{i'}$, if there exists some $a_{i''} \in \mathcal{M}_{i''}$ such that $a_{i''} \overset{p}{\mapsto} a_i$ and $a_{i''} \overset{p}{\mapsto} a_{i'}$.

More precisely, $\overset{p}{\mapsto}$ is a family of relations $(\overset{p}{\mapsto}_{i',i})_{i\leq i'}$, and $a_{i'} \overset{p}{\mapsto} a_i$ is an abbreviation for $a_{i'} \overset{p}{\mapsto}_{i',i} a_i$. The main additional property of a system $(\mathcal{M}_\mathcal{I}, \overset{p}{\mapsto})$ is not transitivity, but rather

$$a_{i'} \asymp a_i \iff a_{i'} \overset{p}{\mapsto} a_i \tag{Factor}$$

for all $i \leq i'$, where the backward implication holds in general.

**Lemma 2.5** *If* (Factor) *holds in a system* $\mathcal{M}_\mathcal{I}$*, then for all* $a_i, c_i \in \mathcal{M}_i$

$$a_i \asymp c_i \iff a_i \equiv c_i \iff a_i \overset{p}{\mapsto} c_i \iff c_i \overset{p}{\mapsto} a_i.$$

To establish (Factor) for the function space, an embedding-projection pair is required. Embeddings enforce the existence of future states of an element and, whenever several such states are available at a later stage, they select one of them in a controlled way. Projections, on the other hand, determine a state at a previous stage, or select a default value if the element did not yet exist at that stage. These families of functions are, like all functions considered here, $\equiv$*-preserving*. That is, a function $f_{i,j} : \mathcal{M}_i \to \mathcal{N}_j$, between sets $\mathcal{M}_i$ and $\mathcal{N}_j$ of systems $\mathcal{M}_\mathcal{I}$ and $\mathcal{N}_\mathcal{J}$, is said to be $\equiv$-preserving if $a_i \equiv c_i$ implies $f(a_i) \equiv f(c_i)$.

Consider a family $emb = (emb_i^{i'})_{i\leq i'}$ of embeddings $emb_i^{i'} : \mathcal{M}_i \to \mathcal{M}_{i'}$, which are $\equiv$-preserving maps satisfying reflexivity and transitivity up to $\equiv$, that is, $emb_i^i(a_i) \equiv a_i$ and $emb_{i'}^{i''}(emb_i^{i'}(a_i)) \equiv emb_i^{i''}(a_i)$. Analogously, let $proj = (proj_i^{i'})_{i\leq i'}$ be a family of projections $proj_i^{i'} : \mathcal{M}_{i'} \to \mathcal{M}_i$.[12] They form a *embedding-projection pair* if $proj_i^{i'}(emb_i^{i'}(a_i)) \equiv a_i$ holds for all $a_i \in \mathcal{M}_i$ and all $i \leq i'$.

If a system admits an embedding–projection pair and some set $\mathcal{M}_i$ is nonempty, then all sets $\mathcal{M}_i$ are nonempty. In other words, the existence of an embedding–projection pair precludes a mixture of inhabited and empty stages. The embeddings $emb$ and projections $proj$ are said to be *coherent* if, for all stages $i \leq i' \leq i''$,

[12] In [15] we called them $\asymp$-embeddings and $\asymp$-projections, however $\equiv$-embeddings and $\equiv$-projections would be better names. Here we simply speak of embeddings and projections.

they satisfy

$$a_{i'} \overset{p}{\mapsto} a_i \implies emb_{i'}^{i''}(a_{i'}) \overset{p}{\mapsto} a_i \qquad \text{for all } a_{i'} \in \mathcal{M}_{i'}, a_i \in \mathcal{M}_i, \tag{Emb}$$

$$a_{i''} \overset{p}{\mapsto} a_i \implies proj_{i''}^{i''}(a_{i''}) \overset{p}{\mapsto} a_i \qquad \text{for all } a_{i''} \in \mathcal{M}_{i''}, a_i \in \mathcal{M}_i. \tag{Proj}$$

Immediate consequences from the coherence conditions are

$$emb_i^{i'}(a_i) \overset{p}{\mapsto} a_i \quad \text{and} \quad a_{i'} \overset{p}{\mapsto} a_i \implies proj_i^{i'}(a_{i'}) \equiv a_i,$$

which state, informally, that the predecessor relation lies between the embeddings and projections.

**Definition 2.6** A *factor system* is a system $(\mathcal{M}_\mathcal{I}, \overset{p}{\mapsto})$ satisfying (Factor), together with a coherent embedding-projection pair $(emb, proj)$. Given any stages $i \leq i'$, and elements $a_i, c_i \in \mathcal{M}_i$ and $a_{i'} \in \mathcal{M}_{i'}$. A factor system $(\mathcal{M}_\mathcal{I}, \overset{p}{\mapsto})$ is called

- *standard* if

$$a_i \equiv c_i \iff a_i = c_i, \tag{Std}$$

- *stable* if the predecessor relations satisfy

$$a_{i'} \overset{p}{\mapsto} a_i \equiv c_i \implies a_{i'} \overset{p}{\mapsto} c_i, \tag{Stab}$$

- *direct* if the embeddings satisfy

$$a_{i'} \overset{p}{\mapsto} a_i \iff a_{i'} \equiv emb_i^{i'}(a_i), \tag{Dir}$$

- *inverse* if the projections satisfy

$$a_{i'} \overset{p}{\mapsto} a_i \iff proj_i^{i'}(a_{i'}) \equiv a_i. \tag{Inv}$$

Note that every standard system is automatically stable and that the factor system is standard precisely when, for every $i \leq i'$, the relation $\overset{p}{\mapsto}_{i',i}$ is a partial function. In every factor system $c_{i'} \equiv a_{i'} \overset{p}{\mapsto} a_i$ entails $c_{i'} \overset{p}{\mapsto} a_i$. To prove certain (though less significant) properties, the "dual" property of stability is required. Stability is a natural requirement, which ensures that $c_{i'} \equiv a_{i'} \overset{p}{\mapsto} a_i \equiv c_i$ implies $c_{i'} \overset{p}{\mapsto} c_i$. Furthermore, all natural examples satisfy this property, and we shall therefore restrict our attention to stable factor systems.

Direct and inverse factor systems possess desirable properties that need not hold in general.

**Lemma 2.7** *Let $\mathcal{M}_\mathcal{I}$ be a factor system, and let $i \leq i' \leq i''$.*

*1. If $\mathcal{M}_\mathcal{I}$ is direct, then $a_{i''} \overset{p}{\mapsto} a_i$ implies $a_{i''} \overset{p}{\mapsto} emb_i^{i'}(a_i)$.*

*2. If $\mathcal{M}_\mathcal{I}$ is inverse, then $a_{i''} \overset{p}{\mapsto} a_{i'}$ implies $a_{i''} \overset{p}{\mapsto} proj_i^{i'}(a_{i'})$.*

The motivation for considering non-standard factor systems is the ability to incorporate equivalence relations and generalizations of setoids. These are important examples in constructive mathematics [27].

*Example 2.8* Each setoid, that is, a set $\mathcal{M}$ equipped with an equivalence relation $\equiv$, gives rise to a factor system with a single stage, where $\overset{p}{\mapsto}$ coincides with $\equiv$. The projection selects a representative from each equivalence class, while the embedding is the identity map. Such a factor system is stable, direct and inverse, and it is standard if and only if $\equiv$ is the identity relation.

A direct factor system is analogous to a direct system, but it uses $\equiv$ in place of equality and includes an additional projection. Inductive definitions provide important examples of direct factor systems.

*Example 2.9* Given a direct system $\mathcal{M}_\mathcal{I}$ of nonempty sets with (strict reflexive and transitive) embeddings $emb_i^{i'}$ for $i \leq i'$. Define the predecessor relation by

$$a_{i'} \overset{p}{\mapsto} a_i \;:\Longleftrightarrow\; emb_{i'}^{i''}(a_{i'}) = emb_i^{i''}(a_i)$$

for some stage $i'' \geq i, i'$. Suppose further that there are distinguished elements $d_i \in \mathcal{M}_i$ for all $i \in \mathcal{I}$ such that $emb_i^{i'}(d_i) \equiv d_{i'}$. Then the projections are defined by

$$proj_i^{i'}(a_{i'}) := \begin{cases} a_i & \text{if } a_{i'} = emb_i^{i'}(a_i) \text{ and } a_i \in \mathcal{M}_i, \\ d_i & \text{otherwise,} \end{cases}$$

so that each element is mapped to the default element at a preceding stage whenever it has no predecessor. The elements $a_i$ in this definition, satisfying $a_{i'} = emb_i^{i'}(a_i)$, are $\equiv$-equivalent, and we assume that one representative is selected; the projections are thus unique only modulo $\equiv$. This construction defines a direct factor system. An important special case arises when the embeddings are injective, so that $\equiv$ is the identity. In this case,

$$a_{i'} \overset{p}{\mapsto} a_i \iff a_{i'} = emb_i^{i'}(a_i)$$

and the projections are uniquely determined. $\mathcal{M}_\mathcal{I}$ is then a standard factor system. Subset inclusions $\mathcal{M}_i \subseteq \mathcal{M}_{i'}$ are a further special case of injective embeddings. Then $a_{i'} \overset{p}{\mapsto} a_i \iff a_{i'} = a_i$, and a single default element $d$ exists in all sets $\mathcal{M}_i$. Examples of this kind include inductive definitions such as the natural numbers — that is, the factor system $(\mathbb{N}_i)_{i \in \mathbb{N}^+}$ discussed in Example 1.1 — as well as the ordinal numbers $(On_i)_{i \in On^+}$, where $On$ denotes the class of ordinals, $On_i = \{a \in On \mid a < i\}$ and $On^+ = On \setminus \{0\}$. In both cases, the default element is 0.

An open-ended type theory provides another instance of the preceding example with injective embeddings.

*Example 2.10* Consider a type theory as open-ended in the sense that both the signature and the type formers may be extended. Let $Typ_{\Sigma;\Gamma}$ denote the set of types in a context $\Gamma$

that can be generated from a signature $\Sigma$ using the permitted type formers (and write $Typ_\Sigma$ when $\Gamma$ is empty). Without loss of generality, assume that *unit* is a type in every signature.

For simplicity, we do not adopt a finitary perspective; instead, we assume that inductively defined totalities are extensionally definite. A finitary treatment would require additional bounds to ensure that the sets of terms remain finite.

Let $\mathcal{I}$ denote the open-ended class of signatures and type formers, with $unit : type \in \Sigma$ for every $\Sigma \in \mathcal{I}$. Without going into details, we assume that there is a directed order $\leq$ on $\mathcal{I}$, defined by $\Sigma_0 \leq \Sigma_1$ if $\Sigma_0$ is contained in $\Sigma_1$. This order is directed because, given any two signatures, one can extend either signature by adding the missing declarations from the other, analogous to the extension (or "weakening") of contexts. Consequently, $(\mathcal{I}, \leq)$ forms an extensional indefinite index class with least element $\Sigma = \{unit : type\}$ (no type formers).

The embeddings are not necessarily subset inclusions. To illustrate, suppose that $\Sigma_1$ extends $\Sigma_0$ by the base type *void*, and let $emb_0^1$ denote the embedding of types from $Typ_{\Sigma_0}$ into $Typ_{\Sigma_1}$. Assume further that there is a dependent type $\langle \mathsf{x} : type\rangle \vdash \varrho$ in $Typ_{\Sigma_0;\langle \mathsf{x}:type\rangle}$. This type is defined for all types in the signature $\Sigma_0$, but not necessarily for the new type *void*. For example, $\varrho$ could be the same as $unit \to \mathsf{x}$ in $\Sigma_0$, but in the presence of the new type *void* we want to embed $\varrho$ as

$$emb_0^1(\langle \mathsf{x} : type\rangle \vdash \varrho) \quad := \quad \langle \mathsf{x} : type\rangle \vdash \mathsf{if}\ \mathsf{x} = void\ \mathsf{then}\ void\ \mathsf{else}\ \varrho.$$

Another situation arises when $\varrho$ is defined recursively, in which case the embedding may require introducing a new case in its definition.

Further examples of factor systems are inverse factor systems. Coinductive definitions provide important instances of these systems.

*Example 2.11* Adopting a finitary perspective, let *Seq* denote the class of in(de)finite binary sequences, given as a system $(Seq_i)_{i\in\mathbb{N}^+}$, where $Seq_i$ consists of sequences of length $i$. The projections *proj* relate a sequence to an initial part, for example, $proj_3^5(01101) = 011$. This makes $(Seq_i)_{i\in\mathbb{N}}$ an inverse system.

Let the predecessor relation $\overset{p}{\mapsto}$ coincide with the projections *proj*. As the system extends, an element such as $011 \in Seq_3$ then differentiates, or splits up, into the sequences 01100, 01101, 01110, 01111 $\in Seq_5$. The embedding, on the other hand, fills missing positions with zeros, e.g., $emb_3^5(011) = 01100$. This makes $(Seq_i)_{i\in\mathbb{N}}$ a factor system, which is standard and inverse.

The following example serves as a preparation for universes, based on the idea that $\mathcal{M}$ consists of increasing classes, with each subset regarded as an approximating state. It mainly demonstrates that one cannot simply use sets and subset inclusion to approximate a class.

*Example 2.12* Let $\mathcal{M}$ be a proper class of sets $a, b, \ldots$, which we may organize as a system $\mathcal{M}_\mathcal{I}$ as follows. The states in $\mathcal{I}$ are all subsets $A, B, \ldots$ of $\mathcal{M}$, ordered by subset inclusion. For each subset $A$, let $\mathcal{M}_A := A$, with the predecessor relation defined by $b \overset{p}{\mapsto} a :\Longleftrightarrow a \subseteq b$.

Then two sets $a, b \in \mathcal{M}$ are consistent, $a \asymp b$, if and only if there exists a set $c \in \mathcal{M}$ such that $a \subseteq c$ and $b \subseteq c$. With these definitions, $\mathcal{M}_\mathcal{I}$ forms a system. However, Condition (Factor) does not hold in general: if $a, b \in \mathcal{M}_A$ and $\{a, b\} \in \mathcal{M}_{A'}$ with $A \subseteq A'$, then $\{a\} \asymp \{b\}$, but not $\{a\} \overset{p}{\mapsto} \{b\}$.

Further examples, as well as the construction of the function space, are provided in [15].

## 2.4 Targets and Limits of a Factor System

Targets and limits $\mathcal{M}$ of a factor system $\mathcal{M}_\mathcal{I}$ are defined analogously to those of direct and inverse systems. A limit is a target satisfying a universal property, and hence limits are unique up to isomorphism. Adding a target $\mathcal{M}$ on top of a system amounts to extending the index class to $\bar{\mathcal{I}} := \mathcal{I} \cup \{top\}$, where $top$ is the greatest element, and setting $\mathcal{M}_{top} := \mathcal{M}$. In particular, a target is a set, not a class. The extended system is again a factor system. Within this extension, let $\overset{p}{\longmapsto}_i$ denote $\overset{p}{\mapsto}_{top,i}$, so that $A \overset{p}{\longmapsto} a_i$ means $A \overset{p}{\mapsto} a_i$ for $A \in \mathcal{M}$; likewise, write $Emb_i$ and $Proj_i$ for $emb_i^{top}$ and $proj_i^{top}$, respectively.

Because the relations $\overset{p}{\mapsto}_{i',i}$ are only partially defined, different to the embeddings or the projections, we require that each element exists at sufficiently many stages. This is expressed by a class of classes $\mathfrak{D}(\mathcal{I}) \subseteq \mathcal{P}(\mathcal{I})$, where $\mathcal{P}(\mathcal{I})$ denotes the class of all subclasses of $\mathcal{I}$. We use the expressions $\mathfrak{D}$-*many* or *indefinitely many* when referring to classes in $\mathfrak{D}(\mathcal{I})$. A target $\mathcal{M}$ — and in particular a limit — of a system $\mathcal{M}_\mathcal{I}$ is therefore defined to satisfy

$$\mathcal{I}_A := \{i \in \mathcal{I} \mid \exists a_i \in \mathcal{M}_i \; A \overset{p}{\longmapsto} a_i\} \in \mathfrak{D}(\mathcal{I}) \tag{1}$$

for all objects $A \in \mathcal{M}$. Thus, a limit element $A$, representing a varying object $a$, has indefinitely many stages $i \in \mathcal{I}$ at which a state $a_i$ of $a$ exists. Extending a system by its limit preserves the essential properties of the original system: if $\mathcal{M}_\mathcal{I}$ is direct (inverse, standard or stable), then the extended system $\mathcal{M}_{\bar{\mathcal{I}}}$ is also direct (inverse, standard or stable respectively). A useful notion of indefinite extensibility depends on the possibility of constructing limits of a system, which in turn depends on the class $\mathfrak{D} := \mathfrak{D}(\mathcal{I})$. To establish the relevant properties[13], the class $\mathfrak{D}$ must form a filter.

**Definition 2.13** An *indefinitely-many filter* is a class of classes $\mathfrak{D} \subseteq \mathcal{P}(\mathcal{I})$ which satisfies

$\mathfrak{D}$ is a (proper) filter on $\{\mathcal{J} \subseteq \mathcal{I} \mid \mathcal{J} \text{ is cofinal}\}$ and $\mathfrak{D}$ contains all up-classes. (Filter)

**Definition 2.14** An *extensible system* is a stable factor system equipped with an indefinitely-many filter.

The following filter can be associated with any index class, and it will be assumed by default whenever no other filter is specified:

$$\mathfrak{D}_{ev}(\mathcal{I}) := \{\mathcal{J} \subseteq \mathcal{I} \mid \uparrow i \subseteq \mathcal{J} \text{ for some } i \in \mathcal{I}\}.$$

[13] The minimal requirement concerning the class of stages $\mathcal{I}_A$ of a limit element $A$ is cofinality. However, in order to compare two elements $A$ and $B$, the intersection $\mathcal{I}_A \cap \mathcal{I}_B$ must itself be at least cofinal. Property (Filter) ensures that this condition is satisfied.

It is the smallest indefinitely-many filter on an index class $\mathcal{I}$, which coincides with the eventuality filter of the net $id : \mathcal{I} \to \mathcal{I}$. In particular, it is the only indefinitely-many filter in the setoid Example 2.8, where $\mathfrak{D}_{ev}(\{\bot\}) = \{\{\bot\}\}$.

In a direct factor system, the class $\mathcal{I}_A$ is an up-class for every limit element $A$. In an inverse factor system, $\mathcal{I}_A$ coincides with the entire index class $\mathcal{I}$. In both cases, for any target element $A \in \mathcal{M}$, the class $\mathcal{I}_A$ is contained in every indefinitely-many filter on $\mathcal{I}$, and in particular in $\mathfrak{D}_{ev}(\mathcal{I})$. In other words, the choice of filter is irrelevant when dealing with direct or inverse factor systems only. For the construction of function spaces, however, the filter plays an essential role. Given two indefinitely-many filter $\mathfrak{D}(\mathcal{I})$ and $\mathfrak{D}(\mathcal{J})$, then

$$\mathcal{H} \in \mathfrak{D}_{\times}(\mathcal{I} \times \mathcal{J}) \;:\Longleftrightarrow\; \exists\, \mathcal{I}' \in \mathfrak{D}(\mathcal{I})\, \exists\, \mathcal{J}' \in \mathfrak{D}(\mathcal{J}) \;\text{ with } \mathcal{I}' \times \mathcal{J}' \subseteq \mathcal{H}$$

defines an indefinitely-many filter. It can be used for the Cartesian product of factor systems; for the function space, however, it is too small. For the function space we employ the filter

$$\mathcal{H} \in \mathfrak{D}_{\to}(\mathcal{I} \times \mathcal{J}) \;:\Longleftrightarrow\; d(\mathcal{H}) \in \mathfrak{D}(\mathcal{I}) \;\text{ with } d(\mathcal{H}) := \{i \in \mathcal{I} \mid \mathcal{H}(i) \in \mathfrak{D}(\mathcal{J})\}, \qquad (2)$$

where $\mathcal{H}(i) := \{j \in \mathcal{J} \mid (i,j) \in \mathcal{H}\}$ for $\mathcal{H} \subseteq \mathcal{I} \times \mathcal{J}$. Intuitively, $d(\mathcal{H})$ is the "relevant domain" of $\mathcal{H}$ when viewed as a multivalued function on indices: each $i \in d(\mathcal{H})$ ensures that $\mathcal{H}(i)$ contains sufficiently many indices. Thus $\mathcal{H}$ belongs to $\mathfrak{D}_{\to}(\mathcal{I} \times \mathcal{J})$ whenever sufficiently many indices lie within its relevant domain.

*Example 2.15* Consider an extensible system of type $nat \to nat$, consisting of functions in $[\mathbb{N}_{\mathbb{N}^+} \to \mathbb{N}_{\mathbb{N}^+}]$, with index type $nat^+ \times nat^+$. If we use the indefinitely-many filter as defined in (2), then every function $f : \mathbb{N} \to \mathbb{N}$ belongs to the system, given by its restrictions $f_{(i,j)} \in [\mathbb{N}_i \to \mathbb{N}_j]$.

If we instead use the smaller indefinitely-many filter $\mathfrak{D}_{\times}(\mathbb{N}^+ \times \mathbb{N}^+)$, which in this case coincides with $\mathfrak{D}_{ev}(\mathbb{N}^+ \times \mathbb{N}^+)$, then only bounded functions $f : \mathbb{N} \to \mathbb{N}$ can belong to the system. The reason is as follows: Suppose there exists a state $f_{(i,j)} \in [\mathbb{N}_i \to \mathbb{N}_j]$ with $(i,j) \in \mathcal{I}_f \in \mathfrak{D}_{ev}(\mathbb{N}^+ \times \mathbb{N}^+)$. Then by definition, $(i',j') \in \mathcal{I}_f$ for all $i' \geq i$ and $j' \geq j$. In particular, this implies that $f_{(i',j)}(a) \in \mathbb{N}_j$ for all $a \in \mathbb{N}_{i'}$ and all $i' \geq i$, so $f$ is bounded above by $j$.

One may construct concrete limits as sets of sets. A set[14] $\alpha \subseteq \bigcup_{i \in \mathcal{I}} \mathcal{M}_i$ in a system $\mathcal{M}_{\mathcal{I}}$ is called a *consistent set* if $a_{i'} \overset{p}{\mapsto} a_i$ holds for all $a_{i'}, a_i \in \alpha$ with $i' \geq i$, and $\mathcal{I}_\alpha \in \mathfrak{D}(\mathcal{I})$, with $\alpha \overset{p}{\longmapsto} a_i \;:\Longleftrightarrow\; a_i \in \alpha$. A *dynamic element* is a consistent set that is maximal with respect to subset inclusion, and we write $\mathcal{EL}(\mathcal{M}_{\mathcal{I}})$ for the set of all dynamic elements of $\mathcal{M}_{\mathcal{I}}$. For each consistent set $\alpha$ there exists a unique dynamic element $\alpha^m$ with $\alpha \subseteq \alpha^m$.

[14] Initially, $\alpha$ is a class, since all object states of a system form a class. However, we select a definite collection of these states, given by the meta-level stage of the investigation. Nevertheless, $\bigcup_{i \in \mathcal{I}} \mathcal{M}_i$ remains a class, and a further subset defined at a later stage may contain elements not in $\alpha$, since such elements did not exist when $\alpha$ was formed. The same applies to the extension $Ext(A)$ defined below.

**Lemma 2.16** *A consistent set $\alpha$ of a factor system is a dynamic element if and only if the following condition holds: for every $a_i$, if there exists $\mathfrak{D}$-many indices $i' \in \mathcal{I}$ and elements $a_{i'} \in \alpha$ with $a_{i'} \overset{p}{\mapsto} a_i$, then $a_i \in \alpha$.*

**Lemma 2.17** *The set $\mathcal{EL}(\mathcal{M}_\mathcal{I})$ forms a limit of a factor system $\mathcal{M}_\mathcal{I}$.*

*Example 2.18* Consider the inverse (factor) system from Example 2.11. In the in(de)finite limit of $(\mathcal{M}_i)_{i\in\mathbb{N}}$, the maximal consistent sets of states are the in(de)finite binary sequences, represented as the class of all their finite initial parts. Structurally, the limit of this factor system is the same as the inverse limit.

If two consistent sets $\alpha$ and $\beta$ are contained in the same dynamic element, we write $\alpha \sim \beta$. The next lemma roughly states that $\alpha \sim \beta$ holds if and only if $\alpha$ and $\beta$ share the same approximations.

**Lemma 2.19** *Let $\alpha$ and $\beta$ be consistent sets of a factor system. Then $\alpha \sim \beta$ if and only if $a_i \equiv b_i$ for all $i \in \mathcal{I}$, $a_i \in \alpha$ and $b_i \in \beta$.*

There is another useful characterization of a limit. To formulate it, define the *extension* of an element $A$ of a target $\mathcal{M}$ as

$$Ext(A) := \{a_i \in \bigcup_{i\in\mathcal{I}} \mathcal{M}_i \mid A \overset{p}{\longmapsto} a_i\}, \tag{3}$$

which is automatically a consistent set. For example, $Ext(\alpha) = \alpha$ for every dynamic element $\alpha \in \mathcal{EL}(\mathcal{M}_\mathcal{I})$. If the system is standard, we may equivalently organize $Ext(A)$ as a family $(a_i)_{i\in\mathcal{I}_A}$.

**Definition 2.20** Given a target $\mathcal{M}$ for a system $\mathcal{M}_\mathcal{I}$. For a consistent set $\alpha$, we call an element $A \in \mathcal{M}$ with $Ext(A) \sim \alpha$ a *limit element* of $\alpha$. We say that:

1. $\mathcal{M}$ is *maximal (over $\mathcal{M}_\mathcal{I}$)* if $Ext(A) \in \mathcal{EL}(\mathcal{M}_\mathcal{I})$ for all $A \in \mathcal{M}$.
2. $\mathcal{M}$ is *extensional (over $\mathcal{M}_\mathcal{I}$)* if $Ext(A) \sim Ext(B)$ implies $A = B$ for all $A, B \in \mathcal{M}$.
3. $\mathcal{M}$ is *complete (over $\mathcal{M}_\mathcal{I}$)* if every consistent set $\alpha$ has a limit element $A \in \mathcal{M}$.

**Lemma 2.21** *A limit of $\mathcal{M}_\mathcal{I}$ is a target that is maximal, extensional and complete over $\mathcal{M}_\mathcal{I}$.*

In other words, each limit element contains all of its possible approximations (maximality), two limit elements coincide if they have the same approximations (extensionality), and for every consistent set of approximations there exists a limit element (completeness).

The material of Section 2.3 and the present section has already been presented in [15] and [11]. We have repeated it here to account for slight modifications and

the introduction of the notion of an extensible system (Definition 2.14), as well as to prepare for Section 3, where these concepts are extended to families of systems.

## 2.5 From Classes to Systems, Syntactically

Before developing the model, we first sketch the syntactic aspect. Details will be left as a future task. For each type $\varrho$ there shall be a *stage type* or *index type* $\varrho^{ix}$ and a *state type* of the system, which could depend on elements of the corresponding index type:

$$\langle \mathsf{i} : \varrho^{ix} \rangle \vdash \varrho^{st}.$$

To each index type there is an associated directed order, described by a term $\mathsf{leq}$, so we may consider for each type $\varrho$ the pair $\langle \varrho^{ix}, \mathsf{leq} \rangle$.

*Example 2.22* The index type *unit* applies to all sets $\mathcal{M} = [\![\varrho]\!]$, regarded as a trivial system $\mathcal{M}_{\{\perp\}}$. From a finitary perspective, the class $\mathbb{N}$ is specified as the system $(\mathbb{N}_i)_{i\in\mathbb{N}^+}$, which corresponds to the translation of the type *nat* into $\langle \mathsf{i} : nat^+ \rangle \vdash nat_{\,\mathsf{i}}$. Similarly, for the case of ordinal numbers, we take $On^{ix} = On^+$. The order on these types are terms denoting, respectively, equality on *unit*, the usual order on the positive natural numbers, and the usual order on the positive ordinal numbers.

Notice that even for a non-dependent base type, the type of the system is a dependent type. But dependent type theory allows base types to depend on terms of other types. So consider, for instance, the type $\langle \mathsf{n} : nat \rangle \vdash vec(\mathsf{n})$, describing the family $(\mathbb{R}^n)_{n\in\mathbb{N}}$. Assume a finitistic perspective in which $\mathbb{N}$ is approximated by $\mathbb{N}_i$, and each space $\mathbb{R}^n$ is approximated by a finite subset of $\mathbb{R}^n$ containing the origin $o \in \mathbb{R}^n$, denoted by $\mathcal{P}^o_{fin}(\mathbb{R}^n)$. Then the system corresponding to the family $(\mathbb{R}^n)_{n\in\mathbb{N}}$ is given by

$$(\mathbb{R}^n_j)_{i\in\mathbb{N}^+,\, n\in\mathbb{N}_i,\, j\in\mathcal{P}^o_{fin}(\mathbb{R}^n)},$$

where each $\mathbb{R}^n_j = j$ is a finite set of vectors containing the origin. Therefore $\langle \mathsf{n} : nat \rangle \vdash vec(\mathsf{n})$ translates to

$$\langle \mathsf{i} : nat^+,\, \mathsf{n} : nat_{\,\mathsf{i}} \rangle \vdash sets^{\circ}_{vec(\mathsf{n})},$$
$$\langle \mathsf{i} : nat^+,\, \mathsf{n} : nat_{\,\mathsf{i}},\, \mathsf{j} : sets^{\circ}_{vec(\mathsf{n})} \rangle \vdash vec(\mathsf{n})_{\,\mathsf{j}},$$

where $vec(\mathsf{n})_{\,\mathsf{j}} = \{\mathsf{x} : vec(\mathsf{n}) \mid \mathsf{x} \in \mathsf{j}\}$ is the type of elements from $\mathbb{R}^n_j$. So whenever we have a dependent type $\langle \mathsf{x} : \varrho \rangle \vdash \sigma$, the associated index type and state type are given, respectively, by

$$\langle \mathsf{i} : \varrho^{ix}, \mathsf{x} : \varrho^{st} \rangle \vdash \sigma^{ix},$$
$$\langle \mathsf{i} : \varrho^{ix}, \mathsf{x} : \varrho^{st},\, \mathsf{j} : \sigma^{ix} \rangle \vdash \sigma^{st}.$$

For the general definition, we extend this translation to typing contexts $\Gamma = \langle \mathsf{x}_0 : \varrho_0, \ldots, \mathsf{x}_{n-1} : \varrho_{n-1} \rangle$. With variables made explicit, the translated context

$\langle \mathsf{x}_0 : \varrho_0, \ldots, \mathsf{x}_{n-1} : \varrho_{n-1} \rangle^\star$ reads as

$$\langle \mathsf{i}_0 : \varrho_0^{ix},\ \mathsf{y}_0 : \varrho_0^{st}, \ldots,\ \mathsf{i}_{n-1} : \varrho_{n-1}^{ix},\ \mathsf{y}_{n-1} : \varrho_{n-1}^{st} \rangle,$$

writing $\mathsf{i}_k$ for $\mathsf{x}_{2k}$ and $\mathsf{y}_k$ for $\mathsf{x}_{2k+1}$. Given as an inductive definition, a dependent type $\Gamma \vdash \varrho$ is translated to

$$\begin{aligned} \Gamma^\star &\vdash \varrho^{ix}, & \langle\rangle^\star &:= \langle\rangle, \\ \Gamma^\star \langle \mathsf{i} : \varrho^{ix} \rangle &\vdash \varrho^{st}, & (\Gamma.\varrho)^\star &:= \Gamma^\star.\varrho^{ix}.\varrho^{st}. \end{aligned}$$

Consider the case of the arrow type in the simply typed $\lambda$-calculus. In this setting, we have

$$(\varrho \to \sigma)^{ix} = \varrho^{ix} \times \sigma^{ix}$$

with product order. This reflects the idea that the function space $[\mathcal{M} \to \mathcal{N}]$, regarded as a system, corresponds to $[\mathcal{M}_i \to \mathcal{N}_j]_{(i,j) \in \mathcal{I} \times \mathcal{J}}$. It is sometimes convenient, however, to employ an equivalent but different type. A simple example is obtained by taking $\varrho^{ix}$ instead of $\varrho^{ix} \times \mathit{unit} = (\varrho \to \mathit{bool})^{ix}$.

*Example 2.23* Using the index and state types from Example 2.22, consider the type $\mathit{seq}$, defined as $\mathit{nat} \to \mathit{nat}_2$. We may assign to $\mathit{seq}$ the index type $\mathit{nat}^+$ instead of $\mathit{nat}^+ \times \mathit{unit}$. As state type then take $\langle \mathsf{i} : \mathit{nat}^+ \rangle \vdash \mathit{seq}_{\mathsf{i}}$.

The subset type $\langle \mathsf{x} : \mathit{seq} \rangle \vdash (\mathit{seq} \to \mathit{bool})_{\,\mathsf{match}}$, introduced in Example 2.1, may then translate to the following index type and state type:

$$\begin{aligned} &\langle \mathsf{i} : \mathit{nat}^+, \mathsf{x} : \mathit{seq}_{\mathsf{i}} \rangle \vdash \mathit{unit}, \\ &\langle \mathsf{i} : \mathit{nat}^+, \mathsf{x} : \mathit{seq}_{\mathsf{i}} \rangle \vdash ((\mathit{seq} \to \mathit{bool})_{\,\mathsf{match}})_{\,\mathsf{i}} := (\mathit{seq}_{\mathsf{i}} \to \mathit{bool})_{\,\mathsf{match}_{\mathsf{i}}}. \end{aligned}$$

Observe that the trivial and unused declaration $\mathsf{j} : \mathit{unit}$ is omitted in the context of the state type.

*Example 2.24* This construction applies whenever $\mathcal{M}$ is a proper class and carries no additional structure. The idea is to take as the index class $\mathcal{I}$ the collection of all (fixed) subsets $\mathcal{P}_{set}(\mathcal{M})$ of $\mathcal{M}$, ordered by subset inclusion, and to define a state $\mathcal{M}_i$ simply as the set $i$. We then regard the class $\mathcal{M} = [\![\varrho]\!]$ as a direct factor system $(\mathcal{M}_i)_{i \in \mathcal{I}}$, as in Example 2.9. However, the empty set must be treated separately, since a factor system does not permit a mixture of inhabited and empty states.

If $\mathcal{M}$ is nonempty, take all subsets of $\mathcal{M}$ that contain a chosen element $d \in \mathcal{M}$, i.e., $\mathcal{I} := \mathcal{P}^d_{set}(\mathcal{M})$, and take $\mathcal{M}_i := i$. Otherwise take $\mathcal{I} := \{\bot\}$ and $\mathcal{M}_\bot := \{*\}$. For a dependent type $\Gamma \vdash \varrho$ this yields the following index and state type:

$$\begin{aligned} \Gamma^\star \vdash \varrho^{ix} &:= \mathsf{if}\ \mathsf{nonEmpty}(\varrho)\ \mathsf{then}\ \mathit{sets}_\varrho^{\mathsf{choose}(\varrho)}\ \mathsf{else}\ \mathit{unit}, \\ \Gamma^\star \langle \mathsf{i} : \varrho^{ix} \rangle \vdash \varrho^{st} &:= \mathsf{if}\ \mathsf{nonEmpty}(\varrho)\ \mathsf{then}\ \mathit{type}_{\mathsf{i}}\ \mathsf{else}\ \mathit{void}. \end{aligned}$$

Since $\varrho$ generally depends on a context $\Gamma$, there may be situations in which $\mathsf{nonEmpty}(\varrho)$ is true and others in which it is false. This occurs in particular when considering possibly proper classes of objects and morphisms in a category. For the type of morphisms $\Gamma \vdash \mathit{hom}(\mathsf{x}_0, \mathsf{x}_1)$ with $\Gamma = \langle \mathsf{x}_0 : \mathit{obj}, \mathsf{x}_1 : \mathit{obj} \rangle$, the corresponding index and state types are

$$\Gamma^\star \vdash \mathit{hom}(\mathsf{x}_0, \mathsf{x}_1)^{ix},$$

$$\Gamma^\star\langle \mathsf{i}_2 : hom(\mathsf{x}_0, \mathsf{x}_1)^{ix}\rangle \vdash hom(\mathsf{x}_0, \mathsf{x}_1)^{st},$$

where $\Gamma^\star = \langle \mathsf{i}_0 : obj^{ix}, \mathsf{y}_0 : obj^{st}, \mathsf{i}_1 : obj^{ix}, \mathsf{y}_1 : obj^{st}\rangle$.

In this section, we have very roughly sketched a translation of a type $\varrho$ into a type of the form $\langle \mathsf{i} : \varrho^{ix}\rangle \vdash \varrho^{st}$ with explicit stages and states. This translation corresponds to the semantic refinement of families of classes to families of systems, developed in the next section.

# 3 Families of Systems

In this section, we make the class-indexed families of classes explicit by describing classes as systems. In Section 2.2, we discussed categorical models and considered class-indexed families of classes as instances of models of dependent type theory. Sections 2.3 and 2.4 introduced extensible systems as a concrete formalization of an expanding class $\mathcal{M}$. We now turn to the situation in which the classes of Section 2.2 are given as (extensible) systems, thereby obtaining system-indexed families of systems. It is convenient for the subsequent sections to assume that each index class $\mathcal{I}$ possesses a least element $\bot$. This assumption is not restrictive for real world examples.

## 3.1 The Index Classes of a Family of Stages

Given a class $\mathcal{N}^A$ depending on an element $A$ of a class $\mathcal{M}$. The system that formalizes the class $\mathcal{N}^A$ will depend on a state $a_i \in \mathcal{M}_i$ of a possibly varying element $a$, represented by $A$. Consequently, the index class of this system also depends on these states $a_i$, so we obtain a family of directed classes

$$\mathcal{J} := (\mathcal{J}^{i,a_i}, \leq_{i,a_i})_{i\in\mathcal{I}, a_i\in\mathcal{M}_i}.$$

We adopt, here and in the following, the convention that the expression $a_i$ already determines its index $i \in \mathcal{I}$. Accordingly, we write $\mathcal{J}^{a_i}$ instead of $\mathcal{J}^{i,a_i}$, and similarly $\leq_{a_i}$ for $\leq_{i,a_i}$. For a specific element $a_i$, we refer to the entire index class $\mathcal{J}^{a_i}$, given implicitly as a sufficiently large state. Our aim is to make this state explicit, which requires specifying a stage $j = \varphi(a_i) \in \mathcal{J}^{a_i}$.

Let us first consider the syntactic representation of the stages of a family of classes, as described in Section 2.5. The implicit interpretation of a dependent type $\langle \mathsf{x} : \varrho\rangle \vdash \sigma$ is given by the dependent product $\Pi \mathsf{x}^\varrho\, \sigma$. The index type associated with $\langle \mathsf{x} : \varrho\rangle \vdash \sigma$ likewise has an implicit reading:

$$\langle \mathsf{i} : \varrho^{ix}, \mathsf{x} : \varrho^{st}\rangle \vdash \sigma^{ix} \qquad \rightsquigarrow \qquad \Sigma\, \mathsf{i}^{\varrho^{ix}}\, \Pi \mathsf{x}^{\varrho^{st}}\, \sigma^{ix}. \tag{4}$$

The semantic treatment requires the use of a subspace $\Pi^{\equiv}_{a_i\in\mathcal{M}_i}\, \mathcal{J}^{a_i}$ of the full product $\Pi_{a_i\in\mathcal{M}_i}\, \mathcal{J}^{a_i}$, in which the relation $\equiv$ is also taken into account. For a given stage $i \in \mathcal{I}$, a function $\varphi \in \Pi_{a_i\in\mathcal{M}_i}\, \mathcal{J}^{a_i}$ is called $\equiv$-preserving if

$$a_i \equiv b_i \implies \varphi(a_i) = \varphi(b_i)$$

for all $a_i, b_i \in \mathcal{M}_i$. The notion of $\equiv$-preservation makes sense if indices are understood to be taken from an implicitly given limit stage at which $\equiv$ is the identity — consequently, there is no need to take $\equiv$ into account for the leading sum, corresponding to the leading $\Sigma\, \mathsf{i}^{\varrho^{ix}}$. We define

$$\Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i} := \{\varphi \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i} \mid \varphi \text{ preserves } \equiv\}.$$

We will use an extensible system $\mathcal{M}_{\mathcal{I}}$ as the base system for a family of systems. This construction eventually requires a relation between the indices of $\mathcal{J}^{a_{i'}}$ and $\mathcal{J}^{a_i}$ whenever $proj_i^{i'}(a_{i'}) \equiv a_i$, for $a_{i'} \in \mathcal{M}_{i'}$ and $a_i \in \mathcal{M}_i$. Since this condition does not presuppose that $a_{i'}$ and $a_i$ represent states of the same element, i.e., that $a_{i'} \overset{p}{\mapsto} a_i$, the corresponding index classes $\mathcal{J}^{a_{i'}}$ and $\mathcal{J}^{a_i}$ may differ completely. To relate these distinct index classes, we therefore introduce additional order relations and embeddings.

The order relation, which we introduce now, may be viewed as a kind of transitive hull of $\overset{p}{\mapsto}$, which lifts $\leq$ from $\mathcal{I}$ to $\mathcal{M}_{\mathcal{I}}$. On $\mathcal{M}_{\mathcal{I}}$ however, no directedness is imposed. Based on this order relation we introduce a family of embeddings on $\mathcal{J}$. This, in turn, enables us to introduce an order relation on $\mathcal{J}$, extending the relations $\leq_{a_i}$, which are defined on a single class $\mathcal{J}^{a_i}$, to a family with possible instances between different classes $\mathcal{J}^{a_i}$ and $\mathcal{J}^{a_{i'}}$. In such cases, a relation $j^{a_i} \leq j^{a_{i'}}$, for $j^{a_i} \in \mathcal{J}^{a_i}$ and $j^{a_{i'}} \in \mathcal{J}^{a_{i'}}$, presupposes that $a_i \leq a_{i'}$.

**Definition 3.1** Let $(\leq_{i,i'})_{i \leq i'}$, $i, i' \in \mathcal{I}$, be a family of orders (reflexive and transitive binary relations) on a system $\mathcal{M}_{\mathcal{I}}$ be given. For brevity, we write $\leq$ instead of $\leq_{i,i'}$. Assume that for all $i \leq i'$ the following conditions hold:

$$a_i \equiv c_i \iff a_i \leq c_i \iff a_i \geq c_i \qquad \text{and} \qquad a_{i'} \overset{p}{\mapsto} a_i \implies a_i \leq a_{i'}. \tag{5}$$

If the base system $\mathcal{M}_{\mathcal{I}}$ is equipped with projections, we define the order relation on $\mathcal{M}_{\mathcal{I}}$ by

$$a_i \leq a_{i'} :\iff proj_i^{i'}(a_{i'}) \equiv a_i. \tag{6}$$

A *stage embedding* on a family of directed classes $\mathcal{J} = (\mathcal{J}^{a_i}, \leq_{a_i})_{i \in \mathcal{I}, a_i \in \mathcal{M}_i}$ is a family of monotone, reflexive and transitive maps

$$smb_{a_i}^{a_{i'}} : \mathcal{J}^{a_i} \to \mathcal{J}^{a_{i'}},$$

defined whenever $a_i \leq a_{i'}$. This stage embedding induces an order $\leq$ on the family $\mathcal{J}$ by

$$j \leq j' :\iff smb_{a_i}^{a_{i'}}(j) \leq_{a_{i'}} j'. \tag{7}$$

In this definition, a more precise notation would explicitly include the indices $i \in \mathcal{I}$, leading to stage embeddings $smb_{i,a_i}^{i',a_{i'}}$. Reflexivity and transitivity of the order relation on $\mathcal{M}_{\mathcal{I}}$ means that $a_i \leq a_i$, and $a_i \leq a_{i'} \leq a_{i''}$ implies $a_i \leq a_{i''}$ whenever $i \leq i' \leq i''$. Reflexivity and transitivity of the stage embeddings says that for all $a_i \leq a_{i'} \leq a_{i''}$,

$$smb_{a_i}^{a_i}(j) = j, \qquad \text{and} \qquad smb_{a_{i'}}^{a_{i''}}(smb_{a_i}^{a_{i'}}(j)) = smb_{a_i}^{a_{i''}}(j). \tag{8}$$

**Definition 3.2** A *family of stages* $(\mathcal{M}_\mathcal{I}, \mathcal{J})$ consists of a family of directed classes $\mathcal{J}$, such that

$$\mathcal{J}^{a_i} = \mathcal{J}^{c_i} \qquad \text{for all } a_i \equiv c_i, i \in \mathcal{I},$$

together with a stage embedding. For $\varphi \in \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$ and $\varphi' \in \Pi^{\equiv}_{a_{i'} \in \mathcal{M}_{i'}} \mathcal{J}^{a_{i'}}$ with $i \leq i'$, define

$$\varphi \leq \varphi' \quad :\Longleftrightarrow \quad (\, a_i \leq a_{i'} \implies \varphi(a_i) \leq \varphi'(a_{i'})\,) \tag{9}$$

for all $a_i \in \mathcal{M}_i$, $a_{i'} \in \mathcal{M}_{i'}$. The class of stages of $(\mathcal{M}_\mathcal{I}, \mathcal{J})$ is then

$$\mathcal{K} := \Sigma_{i \in \mathcal{I}} \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i} \qquad \text{with } (i, \varphi) \leq (i', \varphi') :\Longleftrightarrow i \leq i' \text{ and } \varphi \leq \varphi'.$$

It is straightforward to verify that this order $\leq$ on $\mathcal{K}$ is reflexive. Indeed, for any $a_i \leq c_i$, both in $\mathcal{M}_i$, we have $c_i \equiv a_i$ by (5), and all functions $\varphi \in \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$ preserve $\equiv$. However, transitivity is not immediately evident.

**Lemma 3.3** *Let $(\mathcal{M}_\mathcal{I}, \mathcal{J})$ be a family of stages whose base system $\mathcal{M}_\mathcal{I}$ is an extensible system. Then the order on $\Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$, defined by (9), is transitive.*

*Proof* Assume that $i \leq i' \leq i''$ and $\varphi \leq \varphi' \leq \varphi''$. Let $a_i \leq a_{i''}$, and we must show that $\varphi(a_i) \leq \varphi''(a_{i''})$. By assumption, we have $a_i \equiv proj_i^{i''}(a_{i''})$, and setting $a_{i'} := proj_{i'}^{i''}(a_{i''})$, it follows from (6) that $a_{i'} \leq a_{i''}$. Now we have

$$a_i \equiv proj_i^{i''}(a_{i''}) \equiv proj_i^{i'}(proj_{i'}^{i''}(a_{i''})) = proj_i^{i'}(a_{i'}),$$

and therefore $a_i \leq a_{i'}$. From these inequalities we infer $\varphi(a_i) \leq \varphi'(a_{i'})$ and $\varphi'(a_{i'}) \leq \varphi''(a_{i''})$. By (7) this implies

$$smb_{a_{i'}}^{a_{i''}}(\varphi'(a_{i'})) \leq_{a_{i''}} \varphi''(a_{i''}) \quad \text{and} \quad smb_{a_i}^{a_{i'}}(\varphi(a_i)) \leq_{a_{i'}} \varphi'(a_{i'}).$$

Applying the monotonicity of the stage embeddings to the second inequality gives

$$smb_{a_i}^{a_{i''}}(\varphi(a_i)) = smb_{a_{i'}}^{a_{i''}}(smb_{a_i}^{a_{i'}}(\varphi(a_i))) \leq_{a_{i''}} smb_{a_{i'}}^{a_{i''}}(\varphi'(a_{i'})),$$

where the first equation follows from (8). Thus, by transitivity of $\leq_{a_{i''}}$ on $\mathcal{J}^{a_{i''}}$, we infer that $smb_{a_i}^{a_{i''}}(\varphi(a_i)) \leq_{a_{i''}} \varphi''(a_{i''})$, showing $\varphi(a_i) \leq \varphi''(a_{i''})$. □

**Lemma 3.4** *Let $(\mathcal{M}_\mathcal{I}, \mathcal{J})$ be a family of stages with an extensible base system $\mathcal{M}_\mathcal{I}$. Then the class of stages of $(\mathcal{M}_\mathcal{I}, \mathcal{J})$ is a directed class.*

*Proof* It remains to show directedness. Let $(i, \varphi) \in \Sigma_{i \in \mathcal{I}} \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$ and $(i', \varphi') \in \Sigma_{i' \in \mathcal{I}} \Pi^{\equiv}_{a_{i'} \in \mathcal{M}_{i'}} \mathcal{J}^{a_{i'}}$ be given, then there is some index $i'' \geq i, i'$. Define a function $\varphi'' \in \Pi_{a_{i''} \in \mathcal{M}_{i''}} \mathcal{J}^{a_{i''}}$ by

$$\varphi''(a_{i''}) := j'' \text{ for some } j'' \geq j, j',$$

with $j := smb_{a_i}^{a_{i''}}(\varphi(proj_i^{i''}(a_{i''})))$ and $j' := smb_{a_i}^{a_{i''}}(\varphi'(proj_{i'}^{i''}(a_{i''})))$. The existence of such a stage $j''$ is ensured by the directedness of $\mathcal{J}^{a_{i''}}$. We may take the same $j''$ whenever $j$ and $j'$ coincide, independently from $a_{i''}$. This guarantees that $\varphi''$ preserves $\equiv$, since the projections, the stage embeddings, and the functions $\varphi$ and $\varphi'$ all preserve $\equiv$.

We now verify that $\varphi''$ is an upper bound of $\varphi$ and $\varphi'$. Assume that $a_{i''} \geq a_i$. Then, since $\varphi$ preserves $\equiv$, we have $j = smb_{a_i}^{a_{i''}}(\varphi(a_i))$. By definition $\varphi''(a_{i''}) = j'' \geq j \geq \varphi(a_i)$, where the last inequality holds by (7). This proves $\varphi'' \geq \varphi$, and an analogous argument shows that $\varphi'' \geq \varphi'$. □

We now introduce three running examples that will be further developed later. Example 3.5 illustrates that the fibered structure yields a more general situation than the simple case, even when the same fiber is chosen for all base objects; we refer to this as the *constant family example*. Example 3.6, based on a direct extensible system, encompasses many important cases, including all inductive definitions and the universe discussed in Section 3.6; we call this the *direct system example*. Finally, Example 3.7 uses an inverse factor system as base system, related to coinductive definitions. In this case we adopt a potentialist perspective, assuming that only finite sets are definite, while all infinite collections, such as $\mathbb{N}$, are proper classes. This example is called *inverse system example*.

*Example 3.5* The constant family $(\mathcal{M}_\mathcal{I}, \mathcal{J})$ has the same index class for all elements, that is, $\mathcal{J} := \mathcal{J}^{a_i}$ for all $a_i \in \mathcal{M}_i$, $i \in \mathcal{I}$, with a fixed order $\leq$ on $\mathcal{J}$. Regardless of how the order on $\mathcal{M}_\mathcal{I}$ is defined, the stage embedding is simply a collection of identity maps, where the order defined in (7) coincides with the original order on $\mathcal{J}$. The index class $\Sigma_{i\in\mathcal{I}} \Pi^{\equiv}_{a_i\in\mathcal{M}_i} \mathcal{J}^{a_i}$ thus becomes

$$\Sigma_{i\in\mathcal{I}} [\mathcal{M}_i \to^{\equiv} \mathcal{J}] \;:=\; \{(i,\varphi) \in \Sigma_{i\in\mathcal{I}} [\mathcal{M}_i \to \mathcal{J}] \mid \varphi \text{ preserves } \equiv\}.$$

Even though the index class $\mathcal{J}$ is the same for all $a_i \in \mathcal{M}_i$, the stage $j \in \mathcal{J}^{a_i}$ of an element in the fiber may still depend on the input $a_i$. This differs from the non-fibered function space, where the index class consists of pairs $(i,j) \in \mathcal{I} \times \mathcal{J}$, and a uniform stage $j$ is used for all elements $a_i$.

*Example 3.6* Recall that, by the specific assumption in Section 2.5, all index classes have a least element $\bot$. Take a direct extensible system $\mathcal{M}_\mathcal{I}$ from Example 2.9. We require the fiber index classes to satisfy

$$a_{i'} \overset{p}{\mapsto} a_i \quad \text{implies} \quad \mathcal{J}^{a_{i'}} = \mathcal{J}^{a_i}. \tag{10}$$

If $a_i \equiv proj_i^{i'}(a_{i'})$, then either $a_{i'} \overset{p}{\mapsto} a_i$, in which case $\mathcal{J}^{a_i} = \mathcal{J}^{a_{i'}}$, or $a_i$ is the default value $d_i$, so that $\mathcal{J}^{a_i} = \mathcal{J}^{d_i}$. We define the stage embedding $smb_{a_i}^{a_{i'}}(j) \in \mathcal{J}^{a_{i'}}$ by

$$smb_{a_i}^{a_{i'}}(j) := \begin{cases} j & \text{if } a_{i'} \overset{p}{\mapsto} a_i, \\ \bot & \text{otherwise.} \end{cases}$$

The relation $j \leq j'$ holds either when both indices lie in the same class $\mathcal{J}^{a_i}$ and $j \leq_{a_i} j'$, or when $j \in \mathcal{J}^{d_i}$ for all $i \leq i'$ and $j' \in \mathcal{J}^{a_{i'}}$. In the latter case, $smb_{d_i}^{a_{i'}}(j) = \bot \leq_{a_{i'}} j'$.

Now assume $\mathcal{M}_i \subseteq \mathcal{M}_{i'}$, in which case the condition in (10) is satisfied automatically. Moreover, Condition (9) simplifies to

$$\varphi \leq \varphi' \quad \iff \quad \varphi(a_i) \leq \varphi'(a_i) \text{ for all } a_i \in \mathcal{M}_i, i \in \mathcal{I},$$

because $a_i \leq a_{i'}$ implies that either $a_i = a_{i'}$, or else $a_i$ is the default value $d$. In the latter case, $d \leq a_{i'}$, and $smb_d^{a_{i'}}(\varphi(d)) = \bot \leq_{a_{i'}} \varphi'(a_{i'})$, which always holds.

*Example 3.7* Consider as base system the inverse extensible system from Example 2.11. Here, the order $\geq$ on $(\mathcal{M}_i)_{i\in\mathbb{N}}$ is the same as the relation $\overset{p}{\mapsto}$, which necessitates stage embeddings $smb_{a_i}^{a_{i'}} : \mathcal{J}^{a_i} \to \mathcal{J}^{a_{i'}}$ for all $a_i$, $a_{i'}$ with $a_{i'} \overset{p}{\mapsto} a_i$. In this example, we simply set $\mathcal{J}^{a_i} = \{\bot\}$ for all elements $a_i$, so that the stage embeddings become trivial.

## 3.2 System-indexed Families of Systems

In Section 3.1 we considered the index classes of the fibers, here we examine the full fibered structure. We begin with systems that need not admit an embedding–projection pair.

**Definition 3.8** A *(system-indexed) family of systems* is a pair $(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$ such that $(\mathcal{M}_\mathcal{I}, \mathcal{J})$ is a family of stages over the base system $\mathcal{M}_\mathcal{I}$, and

$$\mathcal{N}_\mathcal{J}^\mathcal{M} := (\mathcal{N}_j^{i,a_i})_{i\in\mathcal{I}, a_i\in\mathcal{M}_i, j\in\mathcal{J}^{a_i}}$$

is a family of sets $\mathcal{N}_j^{i,a_i}$, abbreviated $\mathcal{N}_j^{a_i}$, satisfying

$$\mathcal{N}_j^{a_i} = \mathcal{N}_j^{c_i} \qquad \text{for all } i \in \mathcal{I}, j \in \mathcal{J}^{a_i} \text{ and } a_i \equiv c_i. \tag{11}$$

It is equipped with a family of reflexive predecessor relations $b_{j'} \overset{p}{\mapsto} b_j$ on $\mathcal{N}_\mathcal{J}^\mathcal{M}$, relating elements $b_{j'} \in \mathcal{N}_{j'}^{a_{i'}}$ and $b_j \in \mathcal{N}_j^{a_i}$, whenever $i \leq i'$, $a_i \leq a_{i'}$ and $j \leq j'$.

As for systems, $b_j \equiv d_j$ can be defined as $b_j \overset{p}{\mapsto} d_j$ and $d_j \overset{p}{\mapsto} b_j$ for $b_j, d_j \in \mathcal{N}_j$, and the family is *standard* if $b_j \equiv d_j$ is the same as $b_j = d_j$. Two states $b_j \in \mathcal{N}_j^{a_i}$ and $b_{j'} \in \mathcal{N}_{j'}^{a_{i'}}$ are *consistent*, $b_j \asymp b_{j'}$, if there exists some $b_{j''} \in \mathcal{N}_{j''}^{a_{i''}}$ such that $b_{j''} \overset{p}{\mapsto} b_j$ and $b_{j''} \overset{p}{\mapsto} b_{j'}$.

As a consequence of Definition 3.8, each fiber $((\mathcal{N}_j^{a_i})_{j\in\mathcal{J}^{a_i}}, \overset{p}{\mapsto}_{a_i})$ is itself a system for all $a_i \in \mathcal{M}_i$, $i \in \mathcal{I}$, where $\overset{p}{\mapsto}_{a_i}$ denotes the restriction of $\overset{p}{\mapsto}$ to $(\mathcal{N}_j^{a_i})_{j\in\mathcal{J}^{a_i}}$. Note, however, that relations $b_{j'} \overset{p}{\mapsto} b_j$ on $\mathcal{N}_\mathcal{J}^\mathcal{M}$ may also occur between different fibers over $a_{i'}$ and $a_i$, even when $a_{i'}$ and $a_i$ are not related by the predecessor relation $\overset{p}{\mapsto}$.

*Remark 3.9* A natural strengthening of (10) and (11) is the requirement $\mathcal{J}^{a_{i'}} = \mathcal{J}^{a_i}$ and $\mathcal{N}_j^{a_{i'}} = \mathcal{N}_j^{a_i}$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}^{a_i}$, whenever $a_{i'} \overset{p}{\mapsto} a_i$. Since this condition is not needed in the propositions that follow, we do not adopt it as a general assumption.

In the next step, we consider the embedding-projection pair and thus assume that one exists on the base system $\mathcal{M}_\mathcal{I}$. On the fibers, the embeddings and projections are families of maps

$$emb_{a_i,j}^{a_{i'},j'} : \mathcal{N}_j^{a_i} \to \mathcal{N}_{j'}^{a_{i'}} \qquad \text{and} \qquad proj_{a_i,j}^{a_{i'},j'} : \mathcal{N}_{j'}^{a_{i'}} \to \mathcal{N}_j^{a_i},$$

for $i \leq i'$, $a_i \leq a_{i'}$ and $j \leq j'$, satisfying reflexivity and transitivity up to $\equiv$. Here $emb_{a_i,j}^{a_{i'},j'}$ is itself an abbreviation for $emb_{i,a_i,j}^{i',a_{i'},j'}$ to improve readability. We often suppress $a_i$ and $a_{i'}$ when they are understood, writing simply $emb_j^{j'}$ and $proj_j^{j'}$. However, if $j$ and $j'$ are both the single state $\bot$, we omit this stage and retain $a_i$ and $a_{i'}$.

These embeddings and projections form an embedding-projection pair if for all $j \leq j'$

$$proj_j^{j'}(emb_j^{j'}(b_j)) \equiv b_j.$$

**Definition 3.10** A family of systems $(\mathcal{M}_{\mathcal{I}}, \mathcal{N}_{\mathcal{J}}^{\mathcal{M}})$ equipped with an embedding–projection pair $(emb, proj)$ is called a *(system-indexed) family of extensible systems* if the following conditions hold:

1. $(\mathcal{M}_{\mathcal{I}}, \mathcal{J})$ is a family of stages over an extensible system $\mathcal{M}_{\mathcal{I}}$.
2. For all $b_{j'} \in \mathcal{N}_{j'}^{a_{i'}}$ and $b_j, d_j \in \mathcal{N}_j^{a_i}$ with $a_{i'} \overset{p}{\mapsto} a_i$ and $j \leq j'$,

$$b_{j'} \asymp b_j \iff b_{j'} \overset{p}{\mapsto} b_j, \tag{f-Factor}$$

$$b_{j'} \overset{p}{\mapsto} b_j \equiv d_j \implies b_{j'} \overset{p}{\mapsto} d_j. \tag{f-Stab}$$

3. The pair $(emb, proj)$ is a *coherent* embedding–projection pair on $\mathcal{N}_{\mathcal{J}}^{\mathcal{M}}$, that is, for all $j \leq j' \leq j''$ and for all $b_j \in \mathcal{N}_j^{a_i}$, $b_{j'} \in \mathcal{N}_{j'}^{a_{i'}}$, $b_{j''} \in \mathcal{N}_{j''}^{a_{i''}}$, it satisfies

$$b_{j'} \overset{p}{\mapsto} b_j \implies emb_{j'}^{j''}(b_{j'}) \overset{p}{\mapsto} b_j, \tag{f-Emb}$$

$$b_{j''} \overset{p}{\mapsto} b_j \implies proj_{j'}^{j''}(b_{j''}) \overset{p}{\mapsto} b_j. \tag{f-Proj}$$

4. For each index class $\mathcal{J}^{a_i}$ there exists an indefinitely-many filter.

A family of extensible systems $\mathcal{N}_{\mathcal{J}}^{\mathcal{M}}$ is called *direct* (resp. *inverse*) if, for all $b_j \in \mathcal{N}_j^{a_i}$ and $b_{j'} \in \mathcal{N}_{j'}^{a_{i'}}$ with $j \leq j'$, the following hold:

$$b_{j'} \overset{p}{\mapsto} b_j \iff b_{j'} \equiv emb_j^{j'}(b_j), \tag{f-Dir}$$

$$b_{j'} \overset{p}{\mapsto} b_j \iff proj_j^{j'}(b_{j'}) \equiv b_j. \tag{f-Inv}$$

Similarly to Section 2.3, a consequence of coherence is that

$$emb_j^{j'}(b_j) \overset{p}{\mapsto} b_j \qquad \text{and} \qquad b_{j'} \overset{p}{\mapsto} b_j \implies b_j \equiv proj_j^{j'}(b_{j'}).$$

The following lemma is immediate. It ensures that the interpretation of a dependent type $\langle \mathsf{x} : \varrho \rangle \vdash \mathsf{r} : \sigma$ is an extensible system for each value $a_i$ of type $\varrho$ at stage $i \in \mathcal{I}$.

**Lemma 3.11** *Each fiber system $(\mathcal{N}_j^{a_i})_{j \in \mathcal{J}^{a_i}}$ in a family of extensible systems is an extensible system. The predecessor relation and the embedding-projection pair on the fiber are precisely the restrictions of those of the family.*

We continue with the running examples. Example 3.14, describing a direct extensible system, is closely related to universes.

*Example 3.12* In the constant family example let each fiber be the same system $((\mathcal{N}_j)_{j\in\mathcal{J}}, \overset{p}{\mapsto})$ for all $a_i \in \mathcal{M}_i$. Then the family of (extensible) systems $(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$ can equally be described by the pair $(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J})$ of (extensible) systems. All properties, such as (f-Factor), (f-Stab), and so on, reduce to their counterparts (Factor), (Stab), etc.

A particularly simple instance of this construction occurs when the base system is a singleton set.

**Lemma 3.13** *A family of extensible systems whose base system is a singleton set is isomorphic to a single extensible system.*

*Example 3.14* Consider a direct extensible base system $\mathcal{M}_\mathcal{I}$ with $\mathcal{M}_i \subseteq \mathcal{M}_{i'}$ for all $i \leq i'$ (see Example 3.6). Let $d, e$ be two elements contained in every set $\mathcal{M}_i$. For each $a_i \in \mathcal{M}_i$, suppose we are given an extensible system $(\mathcal{N}_j^{a_i})_{j\in\mathcal{J}^{a_i}}$ as fiber. We further assume that

$$\mathcal{J}^d = \mathcal{J}^e = \{\bot\}, \quad \mathcal{N}_\bot^d = \{*\} \quad \text{and} \quad \mathcal{N}_\bot^a = \emptyset \iff a = e.$$

Thus the fiber over the default element $d \in \mathcal{M}_i$ is a singleton set, and the element $e$ serves as the basis for the (unique) empty fiber. Let $\overset{p}{\mapsto}$ be the union of all predecessor relations on the systems $(\mathcal{N}_j^{a_i})_{j\in\mathcal{J}}$ for all $i \in \mathcal{I}$, $a_i \in \mathcal{M}_i$, augmented by the relations

$$b_j \overset{p}{\mapsto} * \in \mathcal{N}_\bot^d$$

for all $i \in \mathcal{I}$, $a_i \in \mathcal{M}_i$, $j \in \mathcal{J}^{a_i}$, and all $b_j \in \mathcal{N}_j^{a_i}$ such that $d \leq a_i$. This yields a family of systems in the sense of Definition 3.8. Conditions (f-Factor) and (f-Stab) hold because $a_{i'} \overset{p}{\mapsto} a_i$ is the same as $a_{i'} = a_i$, and because each fiber system satisfies (Factor) and (Stab).

The embedding-projection pair on the family is obtained by taking the maps within each individual fiber $\mathcal{N}_j^{a_i}$, together with the additional maps arising from the condition $d \leq a_i$. To define these, let $d_{a_i} \in \mathcal{N}_\bot^{a_i}$ be an arbitrary default element at the initial stage of the fiber over any $a_i \neq e$. For all $j \in \mathcal{J}^{a_i}$ and $d \leq a_i$ (note that $d \not\leq e$), define

$$emb_{d,\bot}^{a_i,j}(*) := emb_\bot^j(d_{a_i}) \qquad \text{and} \qquad proj_{d,\bot}^{a_i,j}(b_j) := *.$$

A straightforward case distinction shows that these definitions yield transitive embeddings and projections, and together they form a coherent embedding-projection pair.

*Example 3.15* We continue the inverse system example; its syntactic presentation already appeared in Example 2.23. For $i > 0$ define

$$\mathcal{N}^{a_i} := \{b_i \in Seq_i \mid (a_i)_k = (b_i)_k \text{ for some } k \in \mathbb{N}_i\},$$

where we suppress the trivial index from $\mathcal{J}^{a_i} = \{\bot\}$ in the notation $\mathcal{N}_j^{a_i}$ and in related constructions.

The predecessor relation $\overset{p}{\mapsto}$ relates any finite sequence to one of its initial parts. Note that elements from different fibers may also be related; for example, $01 \overset{p}{\mapsto} 0$ for $01 \in \mathcal{N}^{00} = \{00, 01, 10\}$ and $0 \in \mathcal{N}^0 = \{0\}$. Define the embeddings by $emb_{a_i}^{a_{i'}}(b_i) := emb_i^{i'}(b_i)$, that is, by adding zeros at the end. The projections are no longer identical to the predecessor relation $\overset{p}{\mapsto}$, instead set

$$proj_{a_i}^{a_{i'}}(b_{i'}) := \begin{cases} proj_i^{i'}(b_{i'}) & \text{if } proj_i^{i'}(b_{i'}) \in \mathcal{N}^{a_i}, \\ a_i & \text{otherwise.} \end{cases}$$

For example let $a_1 = 0$, $a_2 = 01$ and $b_2 = 10$. Then $proj_1^2(10) = 1 \notin \mathcal{N}^0$, so we use the default value, hence $proj_0^{01}(10) = 0$. This defines a family of extensible systems. The properties (f-Factor), (f-Stab), and (f-Emb) follow immediately. For (f-Proj) observe that the "otherwise" case, namely $proj_{i'}^{i''}(b_{i''}) \notin \mathcal{N}^{a_{i'}}$, cannot occur under the assumption $b_{i''} \overset{p}{\mapsto} b_i \in \mathcal{N}^{a_i}$.

## 3.3 The Cartesian Product of a Family of Systems

Recall the variable binding of the index type (4). Constructing the Cartesian product $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$ of a family of extensible systems $(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$ corresponds to the following binding of the stage type (where $\mathcal{M}$ is the interpretation of $\varrho$ and $\mathcal{N}^{a_i}$ the interpretation of $\sigma$ at $a_i$):

$$\langle \mathsf{i} : \varrho^{ix}, \mathsf{x} : \varrho^{st}, \mathsf{j} : \sigma^{ix} \rangle \vdash \sigma^{st} \qquad \rightsquigarrow \qquad \langle \mathsf{j} : \sigma^{ix} \rangle \vdash \Sigma\, \mathsf{i}^{\varrho^{ix}}\, \Pi \mathsf{x}^{\varrho^{st}}\, \sigma^{st}.$$

We begin the semantic description with the case where $(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$ is a family of systems without an embedding-projection pair. The index type of the product is the index type of the underlying family of systems as introduced in Section 3.1.

**Definition 3.16** Let $(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$ be a family of systems over an extensible base system. The class of stages of the Cartesian product $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$ is $\Sigma_{i \in \mathcal{I}} \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$, as specified in Definition 3.2. An element of $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$ at stage $(i, \varphi)$ is a $\equiv$-preserving function $f_{(i,\varphi)} \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}$, written as $f_{(i,\varphi)} \in \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}$, and

$$f_{(i',\varphi')} \overset{p}{\mapsto} f_{(i,\varphi)} :\Longleftrightarrow (\, a_{i'} \overset{p}{\mapsto} a_i \implies f_{(i',\varphi')}(a_{i'}) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i)\,).$$

defines the predecessor relation $\overset{p}{\mapsto}$ on $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$.

The elements of $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$ are therefore functions indexed by stages, written $f_{(i,\varphi)}$, which assign to each $a_i \in \mathcal{M}_i$ an element $f_{(i,\varphi)}(a_i) \in \mathcal{N}^{a_i}_{\varphi(a_i)}$. Each such function is $\equiv$-preserving, in the sense that $a_i \equiv c_i$ implies $f_{(i,\varphi)}(a_i) \equiv f_{(i,\varphi)}(c_i)$ for all $a_i, c_i \in \mathcal{M}_i$. The predecessor relation on the product is defined in direct analogy with the logical relation used in the non-fibered function-space construction. In the special case of a constant family we write $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J})$ instead of $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$. However it is important to note that $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J})$ is not the same as $[\mathcal{M}_\mathcal{I} \to \mathcal{N}_\mathcal{J}]$, as the next example demonstrates.

*Example 3.17* In the case of a constant family, the non-fibered and fibered function spaces differ, because the function-space constructions make use of different classes of stages:

$$\begin{aligned}
[\mathcal{M}_\mathcal{I} \to \mathcal{N}_\mathcal{J}] &:= [\mathcal{M}_i \to^{\equiv} \mathcal{N}_j]_{(i,j) \in \mathcal{I} \times \mathcal{J}}, \\
[\mathcal{M}_i \to^{\equiv} \mathcal{N}_j] &:= \{ f_{(i,j)} : \mathcal{M}_i \to \mathcal{N}_j \mid f_{(i,j)} \text{ preserves } \equiv \}, \\
\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}) &:= (\Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{N}_{\varphi(a_i)})_{(i,\varphi) \in \Sigma_{i \in \mathcal{I}} [\mathcal{M}_i \to^{\equiv} \mathcal{J}]}, \\
\Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{N}_{\varphi(a_i)} &:= \{ f_{(i,\varphi)} \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}_{\varphi(a_i)} \mid f_{(i,\varphi)} \text{ preserves } \equiv \}.
\end{aligned}$$

If $\equiv$ is the identity, then $[\mathcal{M}_i \to^{\equiv} \mathcal{N}_j]$ is the same as the full function space $[\mathcal{M}_i \to \mathcal{N}_j]$, and similarly for $\Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{N}_{\varphi(a_i)}$. If we consider only constant function $\varphi_j : a_i \mapsto j$, then the two function spaces $\Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{N}_j$ and $[\mathcal{M}_i \to^{\equiv} \mathcal{N}_j]$ coincide. Indeed, if we interpret $j$ as an abbreviation for $\varphi_j$, then the function space $[\mathcal{M}_\mathcal{I} \to \mathcal{N}_\mathcal{J}]$ naturally embeds as a subfamily of $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J})$.

As a concrete instance, consider the finitist's function space of natural numbers to themselves, given by $\Pi(\mathbb{N}_\mathbb{N}, \mathbb{N}_\mathbb{N})$ with states $\Pi_{n \in \mathbb{N}_i} \mathbb{N}_{\varphi(n)}$ and index class $\Sigma_{i \in \mathbb{N}}[\mathbb{N}_i \to \mathbb{N}]$. For instance, if $f_{(2,\varphi)} \in \Pi_{n \in \mathbb{N}_2} \mathbb{N}_{\varphi(n)}$ with $\varphi = \{0 \mapsto 5, 1 \mapsto 8\}$, then $f_{(2,\varphi)}(0) \in \mathbb{N}_5$ and $f_{(2,\varphi)}(1) \in \mathbb{N}_8$. Thus, $\Pi(\mathbb{N}_\mathbb{N}, \mathbb{N}_\mathbb{N})$ allows the size of the codomain to depend on the input element, whereas the non-fibered function space $[\mathbb{N}_\mathbb{N} \to \mathbb{N}_\mathbb{N}]$ has a fixed, "uniform" bound $j$ for the codomain.

*Example 3.18* We continue the direct system example from Example 3.14. All functions $f_{(i,\varphi)}$ are $\equiv$-preserving, since $\equiv$ is the identity on $\mathcal{M}_i$. For any $a_i \in \mathcal{M}_i \subseteq \mathcal{M}_{i'}$ we have $f_{(i',\varphi')}(a_i) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i)$ whenever $\varphi \leq \varphi'$. Moreover, if each fiber $(\mathcal{N}^{a_i}_j)_{j \in \mathcal{J}^{a_i}}$ is direct and satisfies $\mathcal{N}^{a_i}_j \subseteq \mathcal{N}^{a_i}_{j'}$ for $j \leq j'$, then it follows that $f_{(i',\varphi')}(a_i) = f_{(i,\varphi)}(a_i)$ for all $a_i \in \mathcal{M}_i$. In that case $f_{(i,\varphi)}$ is simply the restriction of $f_{(i',\varphi')}$ to $\mathcal{M}_i$.

*Example 3.19* In the inverse system example, for any index $i \in \mathcal{I}$, we have $\Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i} = \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}$ at the unique stage $\perp$ of each fiber. The function $\varphi$ is in this example trivial and can be omitted. For $f_{i'} \in \Pi_{a_{i'} \in \mathcal{M}_{i'}} \mathcal{N}^{a_{i'}}$ and $f_i \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}$ we then have $f_{i'} \overset{p}{\mapsto} f_i$ if and only if $f_{i'}(a_{i'}) \overset{p}{\mapsto} f_i(a_i)$ for all $a_{i'} \overset{p}{\mapsto} a_i$.

To establish even basic properties of the dependent product $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$, one already needs an extensible system as base system. Otherwise the index class of the product need not be a directed class. The next lemma is obvious.

**Lemma 3.20** *Given a system-indexed family of systems $(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$, and suppose that $\mathcal{M}_\mathcal{I}$ is an extensible system. Then the Cartesian product $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$ is a system.*

Definition 3.16 introduced the fibered Cartesian product $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$ as a system. In order to make it an extensible system, we present an embedding-projection pair based on those on $\mathcal{M}_\mathcal{I}$ and $\mathcal{N}^\mathcal{M}_\mathcal{J}$. The construction proceeds analogously to the case of the simple function space.

**Definition 3.21** Given a family of extensible systems $(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$, then the embeddings on the Cartesian product $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$ are defined as follows:

$$emb^{(i',\varphi')}_{(i,\varphi)} : \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)} \to \Pi_{a_{i'} \in \mathcal{M}_{i'}} \mathcal{N}^{a_{i'}}_{\varphi'(a_{i'})}$$
$$f_{(i,\varphi)} \mapsto (a_{i'} \mapsto emb^{\varphi'(a_{i'})}_{\varphi(a_i)}(f_{(i,\varphi)}(a_i))), \quad a_i := proj^{i'}_i(a_{i'}).$$

The projections on $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$ are defined accordingly:

$$proj^{(i',\varphi')}_{(i,\varphi)} : \Pi_{a_{i'} \in \mathcal{M}_{i'}} \mathcal{N}^{a_{i'}}_{\varphi'(a_{i'})} \to \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}$$

$$f_{(i',\varphi')} \mapsto (a_i \mapsto proj^{\varphi'(a_{i'})}_{\varphi(a_i)}(f_{(i',\varphi')}(a_{i'}))), \quad a_{i'} := emb^{i'}_i(a_i).$$

The indefinitely-many filter on the fibered function space $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$ will be introduced in Definition 3.28. The following proposition — together with Corollary 3.36 — constitutes the main result of this paper. It extends the corresponding results for ordinary function spaces to their fibered counterparts.

**Proposition 3.22** *Let $(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$ be a family of extensible systems. Then the Cartesian product $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$ is an extensible system. Moreover,*

1. *If each $\mathcal{N}^\mathcal{M}_\mathcal{J}$ is standard, then $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$ is standard.*
2. *If $\mathcal{M}_\mathcal{I}$ is inverse and each $\mathcal{N}^\mathcal{M}_\mathcal{J}$ is direct, then $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$ is direct.*
3. *If $\mathcal{M}_\mathcal{I}$ is direct and each $\mathcal{N}^\mathcal{M}_\mathcal{J}$ is inverse, then $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$ is inverse.*

The proof is lengthy and essentially repeats the argument for non-fibered function spaces. For readability, it has been moved to the appendix.

## 3.4 The Indefinitely-Many Filter of the Product

Having defined the class of stages of the dependent product $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$ as $\Sigma_{i\in\mathcal{I}}\Pi^{\equiv}_{a_i\in\mathcal{M}_i}\mathcal{J}^{a_i}$, we now introduce the indefinitely-many filter on this structure. To this end, it is useful to adopt (2) to the fibered setting. For $i \in \mathcal{I}$, $a_i \in \mathcal{M}_i$, and a subclass $\mathcal{H} \subseteq \Sigma_{i\in\mathcal{I}}\Pi^{\equiv}_{a_i\in\mathcal{M}_i}\mathcal{J}^{a_i}$, define

$$\mathcal{H}(i)(a_i) := \{\varphi(a_i) \in \mathcal{J}^{a_i} \mid \exists \varphi \in \Pi^{\equiv}_{a_i\in\mathcal{M}_i}\mathcal{J}^{a_i}\,(i,\varphi) \in \mathcal{H}\}. \tag{12}$$

For each index $i \in \mathcal{I}$, this yields a function $\mathcal{H}(i) \in \Pi_{a_i\in\mathcal{M}_i}\mathcal{P}(\mathcal{J}^{a_i})$, and hence operations such as $\mathcal{H}(i) \cap \mathcal{H}'(i)$ can be defined pointwise. An immediate consequence is that these functions are $\equiv$-preserving in the following sense:

**Lemma 3.23** *If $a_i \equiv c_i$, then $\mathcal{H}(i)(a_i) = \mathcal{H}(i)(c_i)$ for all $i \in \mathcal{I}$.*

*Proof* Let $j \in \mathcal{H}(i)(a_i)$, witnessed by $(i,\varphi) \in \mathcal{H}$ with $\varphi(c_i) \equiv \varphi(a_i) = j$. By Definition 3.2, $\mathcal{J}^{a_i} = \mathcal{J}^{c_i}$, hence $j \in \mathcal{H}(i)(c_i)$, and the reverse inclusion is symmetric. □

**Lemma 3.24** *Let $i, i' \in \mathcal{I}$ with $i \leq i'$, let $\varphi \in \Pi^{\equiv}_{a_i\in\mathcal{M}_i}\mathcal{J}^{a_i}$ and let $a_{i'} \in \mathcal{M}_{i'}$, then*

$$(\uparrow(i,\varphi))(i')(a_{i'}) = \uparrow smb^{a_{i'}}_{a_i}(\varphi(a_i)), \tag{13}$$

*where $a_i := proj^{i'}_i(a_{i'})$.*

*Proof* To see this, first assume $j'$ lies in the left-hand side. Then there exists $\varphi' \in \Pi^{\equiv}_{a_{i'} \in \mathcal{M}_{i'}} \mathcal{J}^{a_{i'}}$ such that $\varphi \leq \varphi'$ and $j' = \varphi'(a_{i'})$. Since $a_i \leq a_{i'}$, we have $\varphi(a_i) \leq \varphi'(a_{i'}) = j'$ by (9), equivalently $smb^{a_{i'}}_{a_i}(\varphi(a_i)) \leq_{a_{i'}} j'$, so $j'$ lies in the right-hand side.

For the converse direction, suppose $smb^{a_{i'}}_{a_i}(\varphi(a_i)) \leq_{a_{i'}} j'$, i.e., $\varphi(a_i) \leq j'$. Define a function $\varphi' \in \Pi^{\equiv}_{a_{i'} \in \mathcal{M}_{i'}} \mathcal{J}^{a_{i'}}$ by

$$\varphi'(c_{i'}) := \begin{cases} j' & \text{if } a_{i'} \equiv c_{i'}, \\ smb^{c_{i'}}_{c_i}(\varphi(proj^{i'}_i(c_{i'}))) & \text{with } c_i := proj^{i'}_i(a_{i'}) \text{ otherwise.} \end{cases}$$

We aim to show that $\varphi \leq \varphi'$, which implies that $j' = \varphi'(a_{i'})$ lies in the left-hand side. So let $c_i \leq c_{i'}$, which is $c_i \equiv proj^{i'}_i(c_{i'})$ by definition, and we must show $\varphi(c_i) \leq \varphi'(c_{i'})$. First, if $a_{i'} \equiv c_{i'}$, then $\varphi(c_i) = \varphi(a_i) \leq j' = \varphi'(c_{i'})$, because both $\varphi$ and the projections preserves $\equiv$. Otherwise $smb^{c_{i'}}_{c_i}(\varphi(c_i)) = smb^{c_{i'}}_{c_i}(\varphi(proj^{i'}_i(c_{i'}))) = \varphi'(c_{i'})$, so again $\varphi(c_i) \leq \varphi'(c_{i'})$. □

Consider now those subclasses $\mathcal{H} \subseteq \Sigma_{i \in \mathcal{I}} \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$ that satisfy the following condition: for all $i \in \mathcal{I}$ and all $\varphi \in \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$,

$$\forall a_i \in \mathcal{M}_i \ \varphi(a_i) \in \mathcal{H}(i)(a_i) \implies (i, \varphi) \in \mathcal{H}. \tag{14}$$

Loosely speaking, this condition ensures that $\mathcal{H}$ contains all "combinations" of functions $\varphi$ that are already in $\mathcal{H}$. We denote the class of all such subclasses $\mathcal{H}$ by $\mathcal{Q}$:

$$\mathcal{Q} := \{\mathcal{H} \subseteq \Sigma_{i \in \mathcal{I}} \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i} \mid \mathcal{H} \text{ satisfies (14)}\}.$$

*Example 3.25* Given any $i \in \mathcal{I}$ and any $\varphi \in \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$, then

$$\uparrow(i, \varphi) \in \mathcal{Q}. \tag{15}$$

To verify this, fix $i' \in \mathcal{I}$ and let $\mathcal{H} := \uparrow(i, \varphi)$. If $i' \not\geq i$, then $\mathcal{H}(i')(a_{i'}) = \emptyset$ for all $a_{i'} \in \mathcal{M}_{i'}$, so condition (14) is satisfied trivially. Now suppose $i' \geq i$, and let $\varphi' \in \Pi^{\equiv}_{a_{i'} \in \mathcal{M}_{i'}} \mathcal{J}^{a_{i'}}$ be such that $\varphi'(a_{i'}) \in \mathcal{H}(i')(a_{i'})$ for all $a_{i'} \in \mathcal{M}_{i'}$. Condition (14) requires that $(i', \varphi') \in \mathcal{H}$, which means we have to show $\varphi' \geq \varphi$. So let $a_{i'} \geq a_i$, that is, $a_i \equiv proj^{i'}_i(a_{i'})$. Using (13), we obtain $\varphi'(a_{i'}) \geq \varphi(a_i)$, and hence $\varphi' \geq \varphi$.

**Lemma 3.26** *Given $i \in \mathcal{I}$ and $\mathcal{H}, \mathcal{H}' \in \mathcal{Q}$, such that there exists a $\equiv$-preserving section $\psi \in \mathcal{H}(i) \cap \mathcal{H}'(i)$, then*

$$(\mathcal{H} \cap \mathcal{H}')(i) \ = \ \mathcal{H}(i) \cap \mathcal{H}'(i). \tag{16}$$

*Proof* If $\mathcal{M}_i$ is empty, the claim is trivial, so given some $a_i \in \mathcal{M}_i$. The inclusion $\subseteq$ is immediate: if $(i, \varphi) \in \mathcal{H} \cap \mathcal{H}'$ then $\varphi(a_i) \in \mathcal{H}(i)(a_i)$ and $\varphi(a_i) \in \mathcal{H}'(i)(a_i)$ by definition (12).

For the reverse inclusion $\supseteq$ let $j \in \mathcal{H}(i)(a_i) \cap \mathcal{H}'(i)(a_i)$ . We have to provide an index for all elements $c_i \in \mathcal{M}_i$. By hypothesis, there exists $\psi \in \mathcal{H}(i) \cap \mathcal{H}'(i)$, $\equiv$-preserving, so define

$$\varphi(c_i) := \begin{cases} j & \text{if } a_i \equiv c_i, \\ \psi(c_i) & \text{otherwise.} \end{cases}$$

Then $\varphi$ is clearly $\equiv$-preserving. Since $\varphi(c_i) \in \mathcal{H}(i)(c_i)$ for all $c_i$, condition (14) gives $(i, \varphi) \in \mathcal{H}$, and similarly $(i, \varphi) \in \mathcal{H}'$. Hence $j = \varphi(a_i) \in (\mathcal{H} \cap \mathcal{H}')(i)(a_i)$. □

**Lemma 3.27** *The class $\mathcal{Q}$ is closed under intersections.*

*Proof* Let $\mathcal{H}, \mathcal{H}' \in \mathcal{Q}$ and we wish to prove that $\mathcal{H} \cap \mathcal{H}' \in \mathcal{Q}$. Fix $i \in \mathcal{I}$, $\varphi \in \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$, $a_i \in \mathcal{M}_i$, and suppose $\varphi(a_i) \in (\mathcal{H} \cap \mathcal{H}')(i)(a_i)$. By the definition (12), $(\mathcal{H} \cap \mathcal{H}')(i)(a_i) \subseteq \mathcal{H}(i)(a_i) \cap \mathcal{H}'(i)(a_i)$, so $\varphi(a_i) \in \mathcal{H}(i)(a_i)$ and $\varphi(a_i) \in \mathcal{H}'(i)(a_i)$. Consequently $(i, \varphi) \in \mathcal{H}$ and $(i, \varphi) \in \mathcal{H}'$ by (14), which proves $(i, \varphi) \in \mathcal{H} \cap \mathcal{H}'$, and thus $\mathcal{H} \cap \mathcal{H}' \in \mathcal{Q}$. □

**Definition 3.28** Suppose we are given indefinitely-many filters on the index class $\mathcal{I}$ and on each class $\mathcal{J}^{a_i}$. Then the indefinitely-many filter $\mathfrak{D}(\mathcal{K})$ on the class $\mathcal{K} := \Sigma_{i \in \mathcal{I}} \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$ is defined as the upward closure of $\mathfrak{D}_0(\mathcal{K})$, where

$$\begin{aligned} \mathcal{H} \in \mathfrak{D}_0(\mathcal{K}) \ &:\Longleftrightarrow\ \mathcal{H} \in \mathcal{Q} \text{ and } d(\mathcal{H}) \in \mathfrak{D}(\mathcal{I}), \\ d(\mathcal{H}) \ &:= \{i \in \mathcal{I} \mid \mathcal{H}(i)(a_i) \in \mathfrak{D}(\mathcal{J}^{a_i}) \text{ for all } a_i \in \mathcal{M}_i\}. \end{aligned}$$

An immediate consequence of the definition is that

$$\mathcal{H} \in \mathfrak{D}(\mathcal{K}) \implies d(\mathcal{H}) \in \mathfrak{D}(\mathcal{I}). \tag{17}$$

This is due to the fact that $d(.)$ is monotone, that is, $\mathcal{H}_0 \subseteq \mathcal{H}$ implies $d(\mathcal{H}_0) \subseteq d(\mathcal{H})$.

*Example 3.29* Consider the situation of Example 3.5 where the class of stages $\mathcal{J}$ is constant. Assume that $\mathcal{I}' \in \mathfrak{D}(\mathcal{I})$ and $\mathcal{J}' \in \mathfrak{D}(\mathcal{J})$. For each $j \in \mathcal{J}$ let $\varphi_j : \mathcal{M}_i \to \mathcal{J}$ denote a constant function with value $j$, which is clearly $\equiv$-preserving. Define

$$\mathcal{H} := \{(i, \varphi_j) \in \Sigma_{i \in \mathcal{I}'} [\mathcal{M}_i \to^{\equiv} \mathcal{J}] \mid j \in \mathcal{J}'\}.$$

Then $d(\mathcal{H}) = \mathcal{I}' \in \mathfrak{D}(\mathcal{I})$, since for every $i \in \mathcal{I}'$ and every $a_i \in \mathcal{M}_i$ we have $\mathcal{H}(i)(a_i) = \mathcal{J}' \in \mathfrak{D}(\mathcal{J})$. Nevertheless, $\mathcal{H}$ does not necessarily lie in $\mathfrak{D}_0(\mathcal{K})$. Indeed, if both $\mathcal{M}_i$ and $\mathcal{J}'$ contain more than one element, then $\mathcal{H} \notin \mathcal{Q}$: Condition (14) would require that also a non-constant function $\varphi$, say one with $\varphi(a_i) = j$ and $\varphi(c_i) = j'$ for distinct $j, j' \in \mathcal{J}'$, belong to $\mathcal{H}$.

**Lemma 3.30** *The class $\mathfrak{D}_0(\mathcal{K})$ is closed under intersections.*

*Proof* Let $\mathcal{H}, \mathcal{H}' \in \mathfrak{D}_0(\mathcal{K})$ and we have to show that $\mathcal{H} \cap \mathcal{H}' \in \mathfrak{D}_0(\mathcal{K})$. By Lemma 3.27 we already know that $\mathcal{H} \cap \mathcal{H}' \in \mathcal{Q}$. It remains to verify that $d(\mathcal{H} \cap \mathcal{H}') \in \mathfrak{D}(\mathcal{I})$. By assumption there are $\mathfrak{D}$-many stages $i \in \mathcal{I}$ such that $\mathcal{H}(i)(a_i) \in \mathfrak{D}(\mathcal{J}^{a_i})$ for all $a_i \in \mathcal{M}_i$, and the same holds for $\mathcal{H}'$. Hence there are $\mathfrak{D}$-many stages $i$ satisfying both conditions, that is, $\mathcal{H}(i)(a_i) \cap \mathcal{H}'(i)(a_i) \in \mathfrak{D}(\mathcal{J}^{a_i})$.

We now verify the hypothesis of Lemma 3.26 for such $i$: we must exhibit a $\equiv$-preserving section $\psi$ with $\psi(a_i) \in \mathcal{H}(i)(a_i) \cap \mathcal{H}'(i)(a_i)$ for all $a_i \in \mathcal{M}_i$. Since $\mathcal{H}(i)(a_i) \cap \mathcal{H}'(i)(a_i)$ is cofinal in $\mathcal{J}^{a_i}$ (as a consequence of being in $\mathfrak{D}(\mathcal{J}^{a_i})$), and since $\mathcal{J}^{a_i}$ is directed, there exists for each $a_i$ an element $j^{a_i} \in \mathcal{H}(i)(a_i) \cap \mathcal{H}'(i)(a_i)$. So define such a section by $\psi(a_i) := j^{a_i}$, whereby we may choose $\psi$ such that it preserves $\equiv$ (cf. Lemma 3.23).

By Lemma 3.26, $(\mathcal{H} \cap \mathcal{H}')(i)(a_i) = \mathcal{H}(i)(a_i) \cap \mathcal{H}'(i)(a_i) \in \mathfrak{D}(\mathcal{J}^{a_i})$ for all $a_i$. Hence $i \in d(\mathcal{H} \cap \mathcal{H}')$, and since this holds for $\mathfrak{D}$-many $i$, we conclude $d(\mathcal{H} \cap \mathcal{H}') \in \mathfrak{D}(\mathcal{I})$. □

**Lemma 3.31** *The class $\mathfrak{D}(\mathcal{K})$ from Definition 3.28 is an indefinitely-many filter on $\mathcal{K} := \Sigma_{i\in\mathcal{I}}\Pi^{\equiv}_{a_i\in\mathcal{M}_i}\mathcal{J}^{a_i}$.*

*Proof* The class $\mathfrak{D}(\mathcal{K})$ is the upward closure of $\mathfrak{D}_0(\mathcal{K})$, and $\mathfrak{D}_0(\mathcal{K})$ is closed under intersections by Lemma 3.30, so it follows that $\mathfrak{D}(\mathcal{K})$ is a filter.

Next we shall prove that each $\mathcal{H}\in\mathfrak{D}_0(\mathcal{K})$ is cofinal, which implies that each $\mathcal{H}\in\mathfrak{D}(\mathcal{K})$ is cofinal, too. So fix a stage $(i,\varphi)\in\mathcal{K}$. We wish to find some $(i',\varphi')\in\mathcal{H}$ with $(i,\varphi)\leq(i',\varphi')$. There exists a stage $i'\in{\uparrow}i\cap d(\mathcal{H})$, since both classes are in $\mathfrak{D}(\mathcal{I})$. For a similar reason we find, for any $a_{i'}\in\mathcal{M}_{i'}$, a stage $j^{a_{i'}}$ with

$$\varphi'(a_{i'}) := j^{a_{i'}} \in \mathcal{H}(i')(a_{i'}) \ \cap {\uparrow} smb^{a_{i'}}_{a_i}(\varphi(proj^{i'}_i(a_{i'}))). \tag{18}$$

This defines a function $\varphi'\in\Pi^{\equiv}_{a_{i'}\in\mathcal{M}_{i'}}\mathcal{J}^{a_{i'}}$, since we can choose $\varphi'$ in such a way that it preserves $\equiv$. We claim firstly that $(i,\varphi)\leq(i',\varphi')$ and secondly that $(i',\varphi')\in\mathcal{H}$.

For the first claim let $a_i\leq a_{i'}$ and we shall show $\varphi(a_i)\leq\varphi'(a_{i'})$. Now $a_i\leq a_{i'}$ implies $a_i\equiv proj^{i'}_i(a_{i'})$, and without loss of generality, we may assume $a_i = proj^{i'}_i(a_{i'})$. Then (18) immediately shows that $\varphi'(a_{i'})\geq_{a_{i'}} smb^{a_{i'}}_{a_i}(\varphi(a_i))$, which is equivalent to $\varphi'(a_{i'})\geq\varphi(a_i)$. For the second claim we appeal to (14).

Finally, we show that ${\uparrow}(i,\varphi)\in\mathfrak{D}_0(\mathcal{K})$ for all $(i,\varphi)\in\mathcal{K}$, which implies that ${\uparrow}(i,\varphi)\in\mathfrak{D}(\mathcal{K})$. By Lemma 3.24 we know that $({\uparrow}(i,\varphi))(i')(a_{i'})$ is an up-class for each $i'\geq i$ and each $a_{i'}\in\mathcal{M}_{i'}$, and therefore belongs to $\mathfrak{D}(\mathcal{J}^{a_{i'}})$. It follows that ${\uparrow}i\subseteq d({\uparrow}(i,\varphi))$, so $d({\uparrow}(i,\varphi))\in\mathfrak{D}(\mathcal{I})$. Together with (15), this establishes ${\uparrow}(i,\varphi)\in\mathfrak{D}_0(\mathcal{K})$. □

## 3.5 Target Families and Limit Families

Section 2.4 recalled the notion of a target and a limit of a system. In this section we introduce the corresponding notions for families of systems.

**Definition 3.32** Given a target $\mathcal{M}$ for a system $\mathcal{M}_{\mathcal{I}}$. The family $(\mathcal{N}^A)_{A\in\mathcal{M}}$ is called a *target family* for a family of systems $(\mathcal{M}_{\mathcal{I}},\mathcal{N}^{\mathcal{M}}_{\mathcal{J}})$ if, for every $A\in\mathcal{M}$, every $i\in\mathcal{I}$ and every $a_i\in\mathcal{M}_i$,

$$A\overset{p}{\longmapsto}a_i \implies \mathcal{N}^A \text{ is a target for the system } (\mathcal{N}^{a_i}_j)_{j\in\mathcal{J}^{a_i}}.$$

The target family is said to be maximal (extensional, complete) over $\mathcal{N}^{\mathcal{M}}_{\mathcal{J}}$, if each $\mathcal{N}^A$ is maximal (extensional, complete respectively) over $(\mathcal{N}^{a_i}_j)_{j\in\mathcal{J}^{a_i}}$, whenever $A\overset{p}{\longmapsto}a_i$. A family $(\mathcal{N}^A)_{A\in\mathcal{M}}$ is a *limit family* of $(\mathcal{M}_{\mathcal{I}},\mathcal{N}^{\mathcal{M}}_{\mathcal{J}})$ if, for every $A\in\mathcal{M}$, every $i\in\mathcal{I}$ and every $a_i\in\mathcal{M}_i$,

$$A\overset{p}{\longmapsto}a_i \implies \mathcal{N}^A \text{ is a limit of the system } (\mathcal{N}^{a_i}_j)_{j\in\mathcal{J}^{a_i}}.$$

Note that in the definition above, a limit family requires only that $\mathcal{M}$ is a target for the base system $\mathcal{M}_{\mathcal{I}}$, not a limit. Likewise, maximality, extensionality and completeness are demanded solely on the fibers, not on the base system. Note also that the definition does not require base and fiber indices to increase simultaneously, as is typically the case in diagonal constructions. Instead, each target $\mathcal{N}^A$ can be approximated along the index class $\mathcal{J}^{a_i}$ of a single element $a_i$ satisfying $A\overset{p}{\longmapsto}a_i$. Intuitively, this is due to the idea that the relations $\overset{p}{\mapsto}$, and their variants $\overset{p}{\longmapsto}$, are, in essence, an equality.

**Definition 3.33** Given a target family $(\mathcal{N}^A)_{A\in\mathcal{M}}$ for $(\mathcal{M}_\mathcal{I},\mathcal{N}_\mathcal{J}^\mathcal{M})$ and let $\mathcal{K} := \Sigma_{i\in\mathcal{I}}\,\Pi^{\equiv}_{a_i\in\mathcal{M}_i}\,\mathcal{J}^{a_i}$. The $\mathfrak{D}$-*continuous* functions of $\Pi_{A\in\mathcal{M}}\,\mathcal{N}^A$ are defined as the elements of

$$\Pi^{\mathfrak{D}}_{A\in\mathcal{M}}\,\mathcal{N}^A := \{F \in \Pi_{A\in\mathcal{M}}\,\mathcal{N}^A \mid \mathcal{I}_F \in \mathfrak{D}(\mathcal{K})\}.$$

*Example 3.34* The constant family from Example 3.17 gives rise to a pair $(\mathcal{M}_\mathcal{I},\mathcal{N}_\mathcal{J})$. Assume that $\mathcal{M}$ is a target for $\mathcal{M}_\mathcal{I}$ and $\mathcal{N}$ is a target for $\mathcal{N}_\mathcal{J}$. Both the non-fibered function space $[\mathcal{M}_\mathcal{I} \to \mathcal{N}_\mathcal{J}]$ and the fibered function space $\Pi(\mathcal{M}_\mathcal{I},\mathcal{N}_\mathcal{J})$ admit notions of $\mathfrak{D}$-continuous functions, but based on different index classes:

$$[\mathcal{M} \to_\mathfrak{D} \mathcal{N}] := \{F : \mathcal{M} \to \mathcal{N} \mid \mathcal{I}_F \in \mathfrak{D}_\to(\mathcal{I}\times\mathcal{J})\},$$
$$\Pi^{\mathfrak{D}}_{A\in\mathcal{M}}\,\mathcal{N} := \{F : \mathcal{M} \to \mathcal{N} \mid \mathcal{I}_F \in \mathfrak{D}(\Sigma_{i\in\mathcal{I}}\,[\mathcal{M}_i \to^{\equiv} \mathcal{J}])\}.$$

In simple situations these function spaces coincide. For example, take the potentialist point of view on the natural numbers from Example 3.17. Since every finite subset of $\mathbb{N}$ is contained in some set $\mathbb{N}_j$, we obtain $\Pi^{\mathfrak{D}}_{A\in\mathbb{N}}\,\mathbb{N} = [\mathbb{N} \to_\mathfrak{D} \mathbb{N}] = [\mathbb{N} \to \mathbb{N}]$, that is, every function from $\mathbb{N}$ to $\mathbb{N}$ is $\mathfrak{D}$-continuous in both senses.

**Proposition 3.35** *Given a target family $(\mathcal{N}^A)_{A\in\mathcal{M}}$ for a family of extensible systems $(\mathcal{M}_\mathcal{I},\mathcal{N}_\mathcal{J}^\mathcal{M})$.*

1. *The function space $\Pi^{\mathfrak{D}}_{A\in\mathcal{M}}\,\mathcal{N}^A$ is a target for $\Pi(\mathcal{M}_\mathcal{I},\mathcal{N}_\mathcal{J}^\mathcal{M})$.*
2. *The family $(\mathcal{N}^A)_{A\in\mathcal{M}}$ is a limit family of $(\mathcal{M}_\mathcal{I},\mathcal{N}_\mathcal{J}^\mathcal{M})$ if and only if it is maximal, extensional and complete over $\mathcal{N}_\mathcal{J}^\mathcal{M}$.*
3. *If $(\mathcal{N}^A)_{A\in\mathcal{M}}$ is maximal (extensional, complete) over $(\mathcal{M}_\mathcal{I},\mathcal{N}_\mathcal{J}^\mathcal{M})$, then so is $\Pi^{\mathfrak{D}}_{A\in\mathcal{M}}\,\mathcal{N}^A$ over $\Pi(\mathcal{M}_\mathcal{I},\mathcal{N}_\mathcal{J}^\mathcal{M})$.*

*Proof* The assumption that $(\mathcal{N}^A)_{A\in\mathcal{M}}$ is a target family for $(\mathcal{M}_\mathcal{I},\mathcal{N}_\mathcal{J}^\mathcal{M})$ already presupposes that $\mathcal{M}$ is a target for the base system $\mathcal{M}_\mathcal{I}$. The second part then follows immediately from applying Lemma 2.21 pointwise. The proof of the third part is deferred to the appendix. It therefore remains to verify the first part. This amounts to two tasks:

1. Defining the predecessor relation $\overset{p}{\longmapsto}$ and an embedding-projection pair $(Emb, Proj)$ between $\Pi^{\mathfrak{D}}_{A\in\mathcal{M}}\,\mathcal{N}^A$ and the system $\Pi(\mathcal{M}_\mathcal{I},\mathcal{N}_\mathcal{J}^\mathcal{M})$.
2. Show that $\mathcal{I}_F \in \mathfrak{D}(\mathcal{K})$ holds for every $F \in \Pi^{\mathfrak{D}}_{A\in\mathcal{M}}\,\mathcal{N}^A$, where $\mathcal{K} := \Sigma_{i\in\mathcal{I}}\,\Pi^{\equiv}_{a_i\in\mathcal{M}_i}\,\mathcal{J}^{a_i}$.

The second part is trivial due to Definition 3.33. For the first task we define $\overset{p}{\longmapsto}$, $Emb$ and $Proj$ in direct analogy with Definitions 3.16 and 3.21, namely

$$F \overset{p}{\longmapsto} f_{(i,\varphi)} :\iff (\, A \overset{p}{\longmapsto} a_i \implies F(A) \overset{p}{\longmapsto} f_{(i,\varphi)}(a_i)\,),$$
$$Emb_{(i,\varphi)}(f_{(i,\varphi)})(A) := Emb_{\varphi(a_i)}(f_{(i,\varphi)}(a_i)), \qquad a_i := Proj_i(A),$$
$$Proj_{(i,\varphi)}(F)(a_i) := Proj_{\varphi(a_i)}(F(A)), \qquad A := Emb_i(a_i).$$

Consider the extensible system extended by the target. As this is still an extensible system, the required properties follow exactly as in the proof of Proposition 3.22. □

The next corollary follows from Proposition 3.35 together with Lemma 2.21.

**Corollary 3.36** *If $\mathcal{M}$ is a limit of an extensible system $\mathcal{M}_\mathcal{I}$ and $(\mathcal{N}^A)_{A\in\mathcal{M}}$ is a limit family of a family of extensible systems $(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$, then $\Pi^\mathfrak{D}_{A\in\mathcal{M}}\mathcal{N}^A$ is a limit of $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}_\mathcal{J}^\mathcal{M})$.*

## 3.6 An Extensible Universe

This section deals with the interpretation of the type *type*, which is an extensible universe $\mathcal{U}_{\mathcal{I}_u}$ having limit $\mathcal{U}$. The index class of the universe is[15]

$$\mathcal{I}_u := \Sigma_{V\in\mathcal{P}^{\{*\}}_{set}(\mathcal{U})}\,\Pi_{\mathcal{N}\in V}\,\mathcal{N}^{ix},$$
$$(V,\varphi)\leq_u (V',\varphi') :\Longleftrightarrow\ V\subseteq V' \text{ and } \varphi(\mathcal{N})\leq\varphi'(\mathcal{N}) \text{ for all } \mathcal{N}\in V,$$

where the function $\varphi$ assigns to each element $\mathcal{N}\in V$ its stage. This order is directed and has the least element $(\{\{*\}\},\{*\}\mapsto\bot)$. The elements in $\mathcal{U}_{(V,\varphi)}$ are precisely the elements of $V$. Compared to $V$, however, the universe stage $\mathcal{U}_{(V,\varphi)}$ assigns an explicit stage to each of its elements, meaning that $\mathcal{U}_{(V,\varphi)} = \{\mathcal{N}_{\varphi(\mathcal{N})} \mid \mathcal{N}\in V\}$ and $\mathcal{N}\overset{^p}{\longmapsto}\mathcal{N}_{\varphi(\mathcal{N})}$.

For the next definition it is important that a system $\mathcal{N}_\mathcal{J}$ with limit $\mathcal{N}$ as an element of $\mathcal{U}_{\mathcal{I}_u}$ with limit $\mathcal{U}$ is identifiable within this universe by its limit $\mathcal{N}$. The limit $\mathcal{N}$ is itself an element of the limit universe $\mathcal{U}$. The next definition is then a special case of a direct extensible system, introduced in Example 2.9.

**Definition 3.37** Let $\mathcal{U}_{\mathcal{I}_u}$ be a universe with limit $\mathcal{U}$, and let $\mathcal{N}_\mathcal{J}$ be a system with limit $\mathcal{N}$, regarded as an element of the universe. For indices $(V,\varphi)\leq(V',\varphi')$, the predecessor relation $\mathcal{N}_{j'}\overset{p}{\mapsto}\mathcal{N}_j$ on $\mathcal{U}_{\mathcal{I}_u}$ between the states $\mathcal{N}_j\in\mathcal{U}_{(V,\varphi)}$ and $\mathcal{N}_{j'}\in\mathcal{U}_{(V',\varphi')}$ holds, if and only if $j=\varphi(\mathcal{N})$ and $j'=\varphi'(\mathcal{N})$. The embedding and projections are given by

$$emb^{(V',\varphi')}_{(V,\varphi)}(\mathcal{N}_j) := \mathcal{N}_{j'},\qquad proj^{(V',\varphi')}_{(V,\varphi)}(\mathcal{N}_{j'}) := \begin{cases}\mathcal{N}_j & \text{if } \mathcal{N}_j\in\mathcal{U}_{(V,\varphi)},\\ \{*\}_\bot & \text{otherwise.}\end{cases}$$

The indefinitely many filter is the one introduced in Section 3.4.

The universe defined above can increase in two ways, first, new systems may enter the universe, that is, the set $V$ may become larger. Second, the stages of the systems already present may increase as the function $\varphi$ assigns larger stages. But, because the universe is a direct extensible system, it is not possible for a system within it to split into several distinct systems during an extension. Nevertheless, as a system in the universe extends, its elements may split into several states (see Example 2.11), but the system itself does not split up. The class $\mathcal{I}_\mathcal{N}$ at which $\mathcal{N}$ is present, defined by (1), is $\{(V,\varphi)\mid\mathcal{N}\in V\}$. So a single state $\mathcal{N}_j$, of a system $\mathcal{N}_\mathcal{J}$, may appear at many universe stages, namely at all stages $(V,\varphi)$ for which $j=\varphi(\mathcal{N})$ and $\mathcal{N}\in V$.

In general we cannot take $\mathcal{U}$ itself to be one of the sets $V$, since these are fixed sets, whereas the extension of the limit universe $\mathcal{U}$ is context dependent. However, in the

[15] The index class $(\mathcal{I}_u,\leq_u)$ has a syntactic description according to Section 2.5. The class $\mathcal{P}^*_{set}(\mathcal{U})$ is represented by the type $\sigma := sets^{unit}_{type}$ (see Section 2.1 for this notation). The index type of the universe is then given by the pair $\langle\tau_u,\mathsf{leq_u}\rangle$, where $\tau_u := \Sigma\mathsf{v}^\sigma,\Pi\mathsf{x}\in\mathsf{v},\mathsf{x}^{ix}$, and $\mathsf{leq_u}$ is a term representing the order relation $\leq_u$.

special situation that $\mathcal{U}$ does not increase by new classes — such as in Example 3.41 — it suffices to take $\mathcal{U}$ instead of various sets $V$. In this situation, every system $\mathcal{N} \in \mathcal{U}$ is present at all universe stages, and consequently $\mathcal{I}_{\mathcal{N}} = \mathcal{I}_u$ holds for all $\mathcal{N}$.

A universe $\mathcal{U}$ can also be viewed as a family indexed over (the limits of) its elements. Concretely, the base system $\mathcal{M}_{\mathcal{I}}$ has index class $\mathcal{I} = \mathcal{I}_u$ and states $\mathcal{M}_{(V,\varphi)} = V$. The fiber over $\mathcal{N} \in \mathcal{M}_{(V,\varphi)}$ is the set $\mathcal{N}_{\varphi(\mathcal{N})}$. Thus the index class $\mathcal{J}^{(V,\varphi),\mathcal{N}}$ of each fiber is $\{\bot\}$, with $\mathcal{N}_{\bot}^{(V,\varphi),\mathcal{N}} = \mathcal{N}_{\varphi(\mathcal{N})}$. If we add an element $e \in \mathcal{M}_{\bot}$ for an empty type, then this family of extensible systems becomes an instance of Example 3.14.

**Lemma 3.38** *The system $\mathcal{U}_{\mathcal{I}_u}$ is a standard and direct extensible universe. It is the interpretation of the universe type type and has $(\mathcal{I}_u, \leq_u)$ as its index class.*

It is easy to define sub-universes for a given predicate $\Phi : \mathit{type} \to \mathit{bool}$ satisfying $\Phi(\mathit{unit})$ (and $\Phi(\varrho)$ for further types $\varrho$ that should be included). An instance of such a sub-universe is given by nonEmpty, giving rise to the universe of all nonempty classes. Since a universe is a standard system, the extension of an element $\mathcal{N} \in \mathcal{U}$ is, by (3), the family $Ext(\mathcal{N}) = (\mathcal{N}_{\varphi(\mathcal{N})})_{(V,\varphi)\in\mathcal{I}_{\mathcal{N}}}$. The extension of $\mathcal{N}$ is therefore not $\mathcal{N}_{\mathcal{J}}$, except in trivial cases such as $\mathcal{U} = \{\{*\}\}$. In general, the universe $\mathcal{U}_{\mathcal{I}_u}$ contains several "copies" of $\mathcal{N}_j$ at different stages $(V, \varphi)$.

In the following examples we sometimes refer, by slight abuse of notation, to a type $\varrho$ itself as an element of the universe, rather than its interpretation $[\![\varrho]\!]$.

*Example 3.39* The most trivial example of a universe is the one containing only the singleton set $[\![\mathit{unit}]\!] = \{*\}$. A slightly richer example is the universe consisting of the two base types *unit* and *void*, corresponding to the systems $[\![\mathit{unit}]\!]$ and $[\![\mathit{void}]\!] = \emptyset$. We assume that both types are already present at the initial universe stage. Hence the index set of the universe is $\mathcal{I}_u = \{\bot\}$, with

$$\bot := (\{\{*\}, \emptyset\}, \mathcal{N} \mapsto \bot) \qquad \text{and} \qquad \mathcal{U}_{\bot} = \{\{*\}, \emptyset\},$$

where we omitted the unique stage of each system state. Viewed as a family of systems as in Example 3.14, this corresponds to choosing $d = \{*\}$ and $e = \emptyset$.

If the class $Typ_{\Sigma}$ of types generated by the signature $\Sigma$ is a set, then this example generalizes directly to the situation where the universe is a set of sets $V = \{[\![\varrho]\!] \mid \varrho \in Typ_{\Sigma}\}$. For higher types, the set of stages remains a singleton set, for instance $(\mathit{unit} \to \mathit{unit})^{ix} = \{(\bot, \bot)\}$.

Let us turn our attention to composed types. A requirement such as

$$\varphi([\mathcal{M} \to \mathcal{N}]) = (\varphi(\mathcal{M}), \varphi(\mathcal{N})) \in [\mathcal{M} \to \mathcal{N}]^{ix} = \mathcal{M}^{ix} \times \mathcal{N}^{ix}$$

means the following: if $\mathcal{M} \in \mathcal{U}$ and $\mathcal{N} \in \mathcal{U}$ are, at the universe stage $(V, \varphi)$, given at a stage $i$ and $j$, respectively, then the stage of the function space $[\mathcal{M} \to \mathcal{N}] \in \mathcal{U}$ is the pair $(i, j)$. For instance, if $\varphi(\mathbb{N}) = i$, then $\varphi([\mathbb{N} \to \mathbb{N}]) = (i, i)$, so that $[\mathbb{N}_i \to \mathbb{N}_i] \in \mathcal{U}_{(V,\varphi)}$. This is harmless if $\mathbb{N}$ is a completed set, then $i = \bot$ and $\mathbb{N}_i = \mathbb{N}$, but from a finitistic perspective, where $i \in \mathbb{N}^+$, it excludes very basic functions such as

the successor function.[16] Consider now the dependent function space. In this case the corresponding requirement is

$$\varphi(\Pi_{A\in\mathcal{M}}\mathcal{N}^A) = (\varphi(\mathcal{M}), a_i \mapsto \varphi(\mathcal{N}^{Emb_i(a_i)})) \in \Sigma_{i\in\mathcal{M}^{ix}}\,\Pi^{\equiv}_{a_i\in\mathcal{M}_i}\,(\mathcal{N}^{ix})^{a_i}.$$

Thus, if at the universe stage $(V,\varphi)$, $\mathcal{M}\in\mathcal{U}$ is at stage $i=\varphi(\mathcal{M})$, and for each $A\in\mathcal{M}$, $\mathcal{N}^A\in\mathcal{U}$ is at stage $j^A=\varphi(\mathcal{N}^A)$, then the stage of $\Pi_{A\in\mathcal{M}}\mathcal{N}^A\in\mathcal{U}$ is $(i, a_i\mapsto j^{Emb_i(a_i)})$. We will use a stratified version of this condition in Example 3.41, where we consider the situation in which $\mathcal{U}\in\mathcal{U}$.

*Example 3.40* From a finitistic perspective, the universe is at each stage a finite set of finite sets. Suppose that type $nat$ is part of the signature, interpreted as an extensible system with index type $nat^+$. The universe then contains at each stage $(V,\varphi)$ a set $\mathbb{N}_i$ for some $i\in\mathbb{N}^+$. Concretely, this means $\varphi(\{*\})=\bot$ and $\varphi(\mathbb{N})=i$; recall that $\mathbb{N}$ refers to a sufficiently large finite set of natural numbers. If $V=\{\{*\},\mathbb{N},\dots\}$, then the corresponding universe state at $(V,\varphi)$ is $\mathcal{U}_{(V,\varphi)}=\{\{*\},\mathbb{N}_i,\dots\}$. Now consider the dependent type from Section 2.5 $\langle \mathsf{x} : nat\rangle \vdash \mathsf{vec(x)}$. Its index and state type are given by

$$\langle \mathsf{i} : nat^+,\ \mathsf{n} : nat_{\,\mathsf{i}}\rangle \vdash \mathsf{vec(n)}^{ix} \qquad := sets^{\circ}_{\mathsf{vec(n)}},$$

$$\langle \mathsf{i} : nat^+,\ \mathsf{n} : nat_{\,\mathsf{i}},\ \mathsf{j} : sets^{\circ}_{\mathsf{vec(n)}}\rangle \vdash \mathsf{vec(n)}^{st} \qquad := \{\mathsf{x} : \mathsf{vec(n)} \mid \mathsf{x}\in\mathsf{j}\}.$$

The dependent type requires a variable assignment, hence, for each $j\in\mathbb{N}$,

$$\varphi([\![\langle \mathsf{x} : nat\rangle \vdash \mathsf{vec(x)}]\!]_j) \in \mathcal{P}^{o}_{fin}(\mathbb{R}^j).$$

For the dependent function type $\Pi\mathsf{x}^{nat}\,\mathsf{vec(x)}$ we then have

$$(\Pi\mathsf{x}^{nat}\,\mathsf{vec(x)})^{ix} = \Sigma\,\mathsf{i}^{nat^+}\,\Pi\mathsf{x}^{nat_{\,\mathsf{i}}}\,sets^{\circ}_{\mathsf{vec(x)}},$$

$$\langle \mathsf{j} : sets^{\circ}_{vec(\mathsf{x})}\rangle \vdash (\Pi\mathsf{x}^{nat}\,\mathsf{vec(x)})^{st} = \Sigma\,\mathsf{i}^{nat^+}\,\Pi\mathsf{x}^{nat_{\,\mathsf{i}}}\,\mathsf{vec(x)}_{\,\mathsf{j}}.$$

At a universe stage $(V,\varphi)$, the stage of the dependent function type is (here, $Emb_i(n)=n\in\mathbb{N}$ for all $n\in\mathbb{N}_i$):

$$\varphi([\![\Pi\mathsf{x}^{nat}\,\mathsf{vec(x)}]\!]) = \varphi(\Pi_{n\in\mathbb{N}}\,\mathbb{R}^n) \in \Sigma_{i\in\mathbb{N}^+}\,\Pi_{n\in\mathbb{N}_i}\,\mathcal{P}^{o}_{fin}(\mathbb{R}^n).$$

A concrete example of a value of this function $\varphi$ is $\varphi(\Pi_{n\in\mathbb{N}}\,\mathbb{R}^n)=(3,\psi)$ with $\psi(0)=\{0\}$, $\psi(1)=\{0,3.7,\pi\}$ and $\psi(2)=\{(0,0),(\lg 3,e)\}$.

## 3.7 Self-Membered Universes and Universe Levels

In the universe of Example 3.39, all index classes were singleton sets, so $\mathcal{U}$ cannot be an element of itself. Directly applying universe stages to the statement $\mathcal{U}\in\mathcal{U}$ would require that $\varphi(\mathcal{U})=(V,\varphi)$, leading to an infinite regress in the definition of $\varphi$. Hence, describing classes as systems alone does not suffice to describe a universe that contains itself.

This infinite regress is resolved by introducing a hierarchy of universe levels, in the standard way. Intuitively, we start with the outermost (or highest) stage of the universe, $\mathcal{U}$, which encompasses all existing mathematical entities at the current stage. If

[16] The finitistic position is thus more challenging. A weaker condition could be $\varphi([\mathbb{N}\to\mathbb{N}])=(i,j)$ with $i,j\le\varphi(\mathbb{N})$, expressing that the stage of composed types must not exceed the stages of their components (and may use different stages for different occurrences of the same index type). We do not pursue this option here.

we then look inside $\mathcal{U}$ and encounter itself again, what we actually find is a predecessor version of the entire universe $\mathcal{U}$, corresponding to a lower level in the hierarchy.

*Example 3.41* Consider a type theory whose signature $\Sigma$ contains the declaration $type : type$ and whose only type formers are dependent products and subset types. Assume that all base types are non-dependent, and that every base type distinct from *type* is extensionally definite, meaning each is interpreted as a fixed set (a system with an index type *unit*). Let

$$\mathcal{U} = \{[\![\varrho]\!] \mid \varrho \in Typ_\Sigma \text{ a closed type}\},$$

which is a universe that is closed in the sense that no new classes can be added to it, and where each proper class $[\![\varrho]\!]$ contained in $\mathcal{U}$ is given by a sufficiently large stage.

In particular, the expression $\mathcal{U} \in \mathcal{U}$ must be interpreted as follows: at a sufficiently large stage (a notion that is context-dependent), the universe $\mathcal{U}$ on the left-hand side of $\mathcal{U} \in \mathcal{U}$ is an element of the universe $\mathcal{U}$ on the right-hand side, again considered at a sufficiently large stage. These two states, however, are not build within the same context, and their extensions may therefore differ. Indeed, when the stages are made explicit — by using a stratified family of functions $\varphi_n$ indexed by levels $n \in \mathbb{N}$ — it becomes clear that the stages involved in interpreting $\mathcal{U} \in \mathcal{U}$ are distinct.

Because of the restricted set of type formers, every element of the universe is either a subclass of a dependent function space $\Pi_{A\in\mathcal{M}}\,\mathcal{N}^A$, a subclass of the universe itself, or a fixed set. This permits the following recursive definition:

$$\varphi_n(\mathcal{X}) := \begin{cases} (\varphi_n(\mathcal{M}), a_i \mapsto \varphi_n(\mathcal{N}^{Emb_i(a_i)})) & \text{if } \mathcal{X} \subseteq \Pi_{A\in\mathcal{M}}\,\mathcal{N}^A, \\ (\mathcal{U}, \varphi_{n-1}) & \text{if } \mathcal{X} \subseteq \mathcal{U} \text{ and } n > 0, \\ \bot & \text{otherwise.} \end{cases}$$

In this example we have $\mathcal{U} \in \mathcal{P}_{set}(\mathcal{U})$ (because no new elements can occur in the universe) and can thus form

$$\mathcal{U}_n := \mathcal{U}_{(\mathcal{U},\varphi_n)} = \{\mathcal{N}_{\varphi_n(\mathcal{N})} \mid \mathcal{N} \in \mathcal{U}\}$$

with $\mathcal{N} \overset{p}{\longmapsto} \mathcal{N}_{\varphi_n(\mathcal{N})} \in \mathcal{U}_n$. In particular, $\mathcal{U} \overset{p}{\longmapsto} \mathcal{U}_{\varphi_n(\mathcal{U})} = \mathcal{U}_{(\mathcal{U},\varphi_{n-1})} = \mathcal{U}_{n-1} \in \mathcal{U}_n$ for $n > 0$. Fixed sets in $\mathcal{U}$, such as $\mathbb{N}$, satisfy $\varphi_n(\mathbb{N}) = \bot$ and hence occur as elements in all universe states $\mathcal{U}_n$.

Since a reference to $\mathcal{U} \in \mathcal{U}$ reads as $\mathcal{U}_n \in \mathcal{U}_{n+1}$, each such reference requires one further universe level. It is therefore natural to assume, consistent with the standard treatment of universes, that only finitely many occurrences of $\mathcal{U} \in \mathcal{U}$ are used. This entails that, for some sufficiently large index $j \in \mathbb{N}$, the universe $\mathcal{U}_j$ is large enough to serve as the limit universe $\mathcal{U}$.

# 4 Conclusion and Future Work

We have formalized indefinite extensible classes as extensible systems, which support the construction of fibered function spaces and thus provide a model for dependent function types (Π-types). We expect that Σ-types can be handled similarly, and in fact more easily. The model permits the construction of extensible systems that may themselves contain extensible systems, possibly including themselves. This situation usually arises in universes, and the model therefore allows for an interpretation of the judgment $type : type$ that avoids the existence of multiple copies of essentially identical universes.

We can distinguish two layers of the model. The familiar typing judgments form the structural layer, while the dynamic behavior — stages and universe levels — resides "behind" this structural layer. Not every typing judgment necessarily admits an interpretation in extensible systems. Expressions that lack such an interpretation are, for example, those that are "impredicatively inconsistent". The model thus requires some restrictions on the expressiveness of the typing judgments, given by state judgments [11]. On the other hand, an advantage of this additional dynamic layer is that the state judgments provide explicit bounds, which are natural numbers when the stage type is *nat*.[17]

The model is flexible enough to allow mathematicians (or users of proof assistants) to decide which types should be treated as indefinitely extensible. If only the universe type is extensible, the expressive power remains close to standard dependent type theory. At the other extreme, if every type with an infinite domain, such as *nat*, is treated as a proper class, then the model supports a finitistic viewpoint. In that case infinity is understood as potential rather than actual, that is, as an indefinitely extensible finite.

Using this model allows applications in proof assistants such as Rocq, Lean, or Agda. A user may declare specific types $\varrho$ to be indefinitely extensible, which leads the system to request a corresponding index type $\langle \varrho^{ix}, \mathsf{leq} \rangle$ and state type $\langle \mathsf{i} : \varrho^{ix} \rangle \vdash \varrho^{st}$. The universe type *type* is in any case extensible, though users may select an index type different from *nat*. The proof assistant then checks whether each typing judgment has a state judgment and is therefore interpretable in the model. If only the universe is extensible, this checking reduces to an ordinary universe-level constraint. However, if many types are extensible, some terms may fail to have an interpretation and must therefore be reformulated, if possible.[18]

Several important tasks remain for future work: We must present the interpretation of types and terms of a full dependent type theory in this model and prove the soundness of the interpretation. This includes the interpretation of the logical connectives, especially the universal quantifier. The interpretation of the latter cannot be given by a continuous functional and requires the $\ll$-relation from [10]. Universes have been explained conceptually, but their full development remains to be carried out.

Furthermore, the interdependence of stages and elements in dependent type theory does not allow state judgments to generalize straightforwardly. This relates to extending the core fragment — which characterizes a class of terms that can be interpreted

[17] If typing is not decidable, such bounds may not be numerals. So the type theory may become undecidable if arbitrary subset types are used. These bounds may then depend on unresolved mathematical problems. For example, if type *nat* is considered to be indefinite extensible and the set of natural numbers which are the sum of two prime numbers is described by the formula $\Phi$, then a sufficiently large stage of the class $\{i \in \mathbb{N}^+ \mid \Phi(2n+2) \text{ for all } n \in \mathbb{N}^+ \text{ with } 2n+2 < i\}$ will be a variable. If Goldbach's conjecture is true, then each index $i$ is sufficiently large. If $2n+2$, for some $n \in \mathbb{N}$, is not the sum of two primes, then only indices $i > 2n+2$ are sufficiently large.

[18] One example of this is the replacement of functions with relations. So for two indefinitely extensible types $\varrho$ and $\sigma$, variables of type $\varrho \to \sigma$ refer to arbitrary functions. These must be total, i.e., provide outputs independently of all future extensions of the domain and codomain. This is a strong ontological commitment, so that $\varrho \to \sigma$ may not be interpretable, see footnote 1. The situation is different for relations $\varrho \to \sigma \to \mathit{bool}$ together with proofs of totality and right-uniqueness. In a relational representation, the codomain is fixed, and moreover, totality shifts to the epistemic level, becoming something to be proven rather than an a priori semantic requirement.

in extensible systems with limits — from simple type theory [15] to include dependent types.

**Supplementary information.**

# Declarations

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The author declares that he has no conflict of interest.

## Author Contributions

The author is solely responsible for all aspects of the work, including the conception, results, proofs, writing, and revision of the manuscript.

# Appendix A Postponed Proofs

Finally, we present the postponed parts of the proof. Elementary results, such as Lemma 2.5, are used implicitly throughout.

## A.1 Proof of Proposition 3.22

Let the assumptions of Proposition 3.22 hold in the following lemmata, which together suffice to establish the proposition.

**Lemma A.1** *The embeddings on the fibered function space preserve* $\equiv$.

*Proof* Let $f_{(i,\varphi)}, g_{(i,\varphi)} \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}$ with $f_{(i,\varphi)} \equiv g_{(i,\varphi)}$, and let $a_{i'}, c_{i'} \in \mathcal{M}_{i'}$ with $a_{i'} \equiv c_{i'}$. We aim to show that

$$emb^{(i',\varphi')}_{(i,\varphi)}(f_{(i,\varphi)})(a_{i'}) \equiv emb^{(i',\varphi')}_{(i,\varphi)}(g_{(i,\varphi)})(c_{i'}). \tag{A1}$$

Since projections preserve $\equiv$, it follows that $proj^{i'}_i(a_{i'}) \equiv proj^{i'}_i(c_{i'})$, and hence $f_{(i,\varphi)}(proj^{i'}_i(a_{i'})) \equiv g_{(i,\varphi)}(proj^{i'}_i(c_{i'}))$. As embeddings preserve $\equiv$, we obtain

$$emb^{(i',\varphi')}_{(i,\varphi)}(f_{(i,\varphi)})(a_{i'}) = emb^{j'}_j\,(f_{(i,\varphi)}(proj^{i'}_i(a_{i'}))) \equiv$$
$$emb^{j'}_j\,(g_{(i,\varphi)}(proj^{i'}_i(c_{i'}))) = emb^{(i',\varphi')}_{(i,\varphi)}(g_{(i,\varphi)})(c_{i'}),$$

where $j := \varphi(proj^{i'}_i(a_{i'})) = \varphi(proj^{i'}_i(c_{i'}))$ and $j' := \varphi'(a_{i'}) = \varphi'(c_{i'})$, establishing (A1). □

**Lemma A.2** *The embeddings on the fibered function space are reflexive modulo* $\equiv$.

*Proof* The claim is equivalent to

$$emb^{(i,\varphi)}_{(i,\varphi)}(f_{(i,\varphi)}) \equiv f_{(i,\varphi)} \quad \text{for} \quad f_{(i,\varphi)} \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}.$$

Assume $a_i \equiv c_i$ holds for $a_i, c_i \in \mathcal{M}_i$. We have to show $emb^{(i,\varphi)}_{(i,\varphi)}(f_{(i,\varphi)})(a_i) \equiv f_{(i,\varphi)}(c_i)$. From $proj^i_i(a_i) \equiv a_i \equiv c_i$ we infer that $f_{(i,\varphi)}(proj^i_i(a_i)) \equiv f_{(i,\varphi)}(c_i)$, since $f_{(i,\varphi)}$ preserves $\equiv$. Moreover, $j := \varphi(proj^i_i(a_i)) = \varphi(a_i)$, since $proj^i_i(a_i) \equiv a_i$ and $\varphi$ preserves $\equiv$. From this, we deduce that

$$emb^{(i,\varphi)}_{(i,\varphi)}(f_{(i,\varphi)})(a_i) = emb^j_j(f_{(i,\varphi)}(proj^i_i(a_i))) \equiv f_{(i,\varphi)}(proj^i_i(a_i)) \equiv f_{(i,\varphi)}(c_i),$$

and hence $emb^{(i,\varphi)}_{(i,\varphi)}(f_{(i,\varphi)})(a_i) \equiv f_{(i,\varphi)}(c_i)$. □

**Lemma A.3** *The embeddings on the fibered function space are transitive modulo* $\equiv$.

*Proof* We shall prove

$$emb^{(i'',\varphi'')}_{(i',\varphi')}(emb^{(i',\varphi')}_{(i,\varphi)}(f_{(i,\varphi)})) \equiv emb^{(i'',\varphi'')}_{(i,\varphi)}(f_{(i,\varphi)})$$

for $(i,\varphi) \leq (i',\varphi') \leq (i'',\varphi'')$ and $f_{(i,\varphi)} \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}$. Let $a_{i''} \equiv c_{i''}$. With $a_{i'} := proj^{i''}_{i'}(a_{i''})$, $a_i := proj^{i'}_i(a_{i'})$ and $c_i := proj^{i''}_i(c_{i''})$ we have $a_i \equiv c_i$, since projections are transitive and preserve $\equiv$. This results in $j := \varphi(a_i) = \varphi(c_i)$, $j' := \varphi'(a_{i'})$ and $j'' := \varphi''(a_{i''}) = \varphi''(c_{i''})$, because $\varphi$ and $\varphi''$ preserve $\equiv$. Our aim is to show

$$emb^{j''}_{j'}(emb^{j'}_j(f_{(i,\varphi)}(a_i))) \equiv emb^{j''}_j(f_{(i,\varphi)}(c_i)). \tag{A2}$$

We have $a_i \equiv proj^{i''}_i(a_{i''}) \equiv proj^{i''}_i(c_{i''}) = c_i$. Moreover, $f_{(i,\varphi)}$ and $emb^{j''}_j$ both preserve $\equiv$, which entails

$$emb^{j''}_{j'}(emb^{j'}_j(f_{(i,\varphi)}(a_i))) \equiv emb^{j''}_j(f_{(i,\varphi)}(a_i)) \equiv emb^{j''}_j(f_{(i,\varphi)}(c_i)).$$

This establishes (A2). □

Similar statements and proofs hold for the projections on the fibered function space. Moreover, it is straightforward to show that projections and embeddings form an embedding-projection pair.

**Lemma A.4** *The fibered function space satisfies conditions* (Factor) *and* (Stab).

*Proof* Given functions $f_{(i,\varphi)} \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}$, $f_{(i',\varphi')} \in \Pi_{a_{i'} \in \mathcal{M}_{i'}} \mathcal{N}^{a_{i'}}_{\varphi'(a_{i'})}$ and $f_{(i'',\varphi'')} \in \Pi_{a_{i''} \in \mathcal{M}_{i''}} \mathcal{N}^{a_{i''}}_{\varphi''(a_{i''})}$ with $(i,\varphi) \leq (i',\varphi') \leq (i'',\varphi'')$.

We begin with Condition Factor. Assume that $f_{(i'',\varphi'')} \overset{p}{\mapsto} f_{(i',\varphi')}$, $f_{(i'',\varphi'')} \overset{p}{\mapsto} f_{(i,\varphi)}$ and $a_{i'} \overset{p}{\mapsto} a_i$. We need to show

$$\mathcal{N}^{a_{i'}}_{\varphi'(a_{i'})} \ni f_{(i',\varphi')}(a_{i'}) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i) \in \mathcal{N}^{a_i}_{\varphi(a_i)}.$$

Since $a_{i'} \overset{p}{\mapsto} a_i$, Condition (Emb) implies $emb_{i'}^{i''}(a_{i'}) \overset{p}{\mapsto} a_i$, and therefore $f_{(i'',\varphi'')}(emb_{i'}^{i''}(a_{i'})) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i)$. On the other hand, $emb_{i'}^{i''}(a_{i'}) \overset{p}{\mapsto} a_{i'}$ implies $f_{(i'',\varphi'')}(emb_{i'}^{i''}(a_{i'})) \overset{p}{\mapsto} f_{(i',\varphi')}(a_{i'})$, so that $f_{(i',\varphi')}(a_{i'}) \asymp f_{(i,\varphi)}(a_i)$. Condition (f-Factor), together with $a_{i'} \overset{p}{\mapsto} a_i$ and $\varphi(a_i) \leq \varphi'(a_{i'})$, then yields $f_{(i',\varphi')}(a_{i'}) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i)$, as required.

For (Stab) assume that $f_{(i',\varphi')} \overset{p}{\mapsto} f_{(i,\varphi)} \asymp g_{(i,\varphi)}$ with $g_{(i,\varphi)} \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}$. Then, by (Factor), we have $f_{(i,\varphi)} \overset{p}{\mapsto} g_{(i,\varphi)}$. We have to show that $f_{(i',\varphi')} \overset{p}{\mapsto} g_{(i,\varphi)}$. First, note that reflexivity $a_i \overset{p}{\mapsto} a_i$ implies $f_{(i,\varphi)}(a_i) \overset{p}{\mapsto} g_{(i,\varphi)}(a_i)$. Next, let $a_{i'} \overset{p}{\mapsto} a_i$. In order to establish $f_{(i',\varphi')}(a_{i'}) \overset{p}{\mapsto} g_{(i,\varphi)}(a_i)$, we apply Condition (f-Stab) to $f_{(i',\varphi')}(a_{i'}) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i) \asymp g_{(i,\varphi)}(a_i)$. □

**Lemma A.5** *The embeddings on the fibered function space satisfy Condition* (Emb).

*Proof* Let $f_{(i',\varphi')} \overset{p}{\mapsto} f_{(i,\varphi)}$ and $a_{i''} \overset{p}{\mapsto} a_i$. We aim to show that

$$\mathcal{N}^{a_{i''}}_{\varphi''(a_{i''})} \ni emb^{(i'',\varphi'')}_{(i',\varphi')}(f_{(i',\varphi')})(a_{i''}) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i) \in \mathcal{N}^{a_i}_{\varphi(a_i)}.$$

Since $a_{i''} \overset{p}{\mapsto} a_i$, Condition (Proj) yields $a_{i'} \overset{p}{\mapsto} a_i$, where $a_{i'} := proj_{i'}^{i''}(a_{i''})$. Hence $f_{(i',\varphi')}(a_{i'}) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i)$. By Condition (f-Emb) we obtain

$$emb^{(i'',\varphi'')}_{(i',\varphi')}(f_{(i',\varphi')})(a_{i''}) = emb_{j'}^{j''}(f_{(i',\varphi')}(a_{i'})) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i)$$

where $j' := \varphi'(a_{i'})$ and $j'' := \varphi''(a_{i''})$. Then $j' \leq j''$ follows from $\varphi' \leq \varphi''$ and $a_{i'} \leq a_{i''}$, the latter being a consequence of $proj_{i'}^{i''}(a_{i''}) \equiv a_{i'}$. □

**Lemma A.6** *The projections on the fibered function space satisfy Condition* (Proj).

*Proof* Let $f_{(i'',\varphi'')} \overset{p}{\mapsto} f_{(i,\varphi)}$ and $a_{i'} \overset{p}{\mapsto} a_i$, and we wish to show

$$\mathcal{N}^{a_{i'}}_{\varphi'(a_{i'})} \ni proj^{(i'',\varphi'')}_{(i',\varphi')}(f_{(i'',\varphi'')})(a_{i'}) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i) \in \mathcal{N}^{a_i}_{\varphi(a_i)}.$$

Condition (Emb), together with $a_{i'} \overset{p}{\mapsto} a_i$, ensures that $a_{i''} \overset{p}{\mapsto} a_i$, where $a_{i''} := emb_{i'}^{i''}(a_{i'})$. Hence $f_{(i'',\varphi'')}(a_{i''}) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i)$. By Condition (f-Proj), we conclude that

$$proj^{(i'',\varphi'')}_{(i',\varphi')}(f_{(i'',\varphi'')})(a_{i'}) = proj_{j'}^{j''}(f_{(i'',\varphi'')}(a_{i''})) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i)$$

where $j' := \varphi'(a_{i'})$ and $j'' := \varphi''(a_{i''})$. Note that $j' \leq j''$ follows from $\varphi' \leq \varphi''$ together with $a_{i'} \leq a_{i''}$, the latter being a consequence of $proj_{i'}^{i''}(a_{i''}) = proj_{i'}^{i''}(emb_{i'}^{i''}(a_{i'})) \equiv a_{i'}$. □

**Lemma A.7** *If each* $\mathcal{N}^{\mathcal{M}}_{\mathcal{J}}$ *is standard, so is* $\Pi(\mathcal{M}_{\mathcal{I}}, \mathcal{N}^{\mathcal{M}}_{\mathcal{J}})$.

*Proof* Let $f_{(i,\varphi)} \equiv g_{(i,\varphi)}$, both in $\Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}$. For all $a_i \in \mathcal{M}_i$ we have $a_i \equiv a_i$, hence $f_{(i,\varphi)}(a_i) \equiv g_{(i,\varphi)}(a_i)$. This gives $f_{(i,\varphi)}(a_i) = g_{(i,\varphi)}(a_i)$ and as a consequence $f_{(i,\varphi)} = g_{(i,\varphi)}$. □

**Lemma A.8** *If $\mathcal{M}_{\mathcal{I}}$ is inverse and $\mathcal{N}_{\mathcal{J}}^{\mathcal{M}}$ direct, then $\Pi(\mathcal{M}_{\mathcal{I}}, \mathcal{N}_{\mathcal{J}}^{\mathcal{M}})$ is direct.*

*Proof* For the $\Longrightarrow$-part of (Dir) assume that $f_{(i',\varphi')} \overset{p}{\mapsto} f_{(i,\varphi)}$ with $f_{(i,\varphi)} \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}$, $f_{(i',\varphi')} \in \Pi_{a_{i'} \in \mathcal{M}_{i'}} \mathcal{N}^{a_{i'}}_{\varphi'(a_{i'})}$ and $(i,\varphi) \leq (i',\varphi')$. We aim to show that $f_{(i',\varphi')} \equiv emb^{(i',\varphi')}_{(i,\varphi)}(f_{(i,\varphi)})$. Let $a_{i'} \equiv c_{i'}$; we must prove that

$$f_{(i',\varphi')}(a_{i'}) \equiv emb^{(i',\varphi')}_{(i,\varphi)}(f_{(i,\varphi)})(c_{i'}) = emb^{j'}_{j}(f_{(i,\varphi)}(c_i))$$

where $j' := \varphi'(a_{i'})$, $j := \varphi(c_i)$ and $c_i := proj^{i'}_{i}(c_{i'})$. Since projections preserve $\equiv$, we have $proj^{i'}_{i}(a_{i'}) \equiv c_i$. Because $\mathcal{M}_{\mathcal{I}}$ is inverse, it follows that $a_{i'} \overset{p}{\mapsto} c_i$, and hence $f_{(i',\varphi')}(a_{i'}) \overset{p}{\mapsto} f_{(i,\varphi)}(c_i)$. Consequently, since $\mathcal{N}_{\mathcal{J}}^{\mathcal{M}}$ is direct, we obtain $f_{(i',\varphi')}(a_{i'}) \equiv emb^{j'}_{j}(f_{(i,\varphi)}(c_i))$.

For the converse direction assume $f_{(i',\varphi')} \equiv emb^{(i',\varphi')}_{(i,\varphi)}(f_{(i,\varphi)})$. This entails

$$f_{(i',\varphi')}(a_{i'}) \equiv emb^{(i',\varphi')}_{(i,\varphi)}(f_{(i,\varphi)})(a_{i'}) = emb^{j'}_{j}(f_{(i,\varphi)}(proj^{i'}_{i}(a_{i'}))), \tag{A3}$$

where $j' := \varphi'(a_{i'})$ and $j := \varphi(proj^{i'}_{i}(a_{i'}))$. Let $a_{i'} \overset{p}{\mapsto} a_i$, and we aim to prove that $f_{(i',\varphi')}(a_{i'}) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i)$. From $a_{i'} \overset{p}{\mapsto} a_i$ we obtain $proj^{i'}_{i}(a_{i'}) \equiv a_i$. Now, $f_{(i,\varphi)}$, $\varphi$ and the embeddings preserve $\equiv$, hence $emb^{j'}_{j}(f_{(i,\varphi)}(proj^{i'}_{i}(a_{i'}))) \equiv emb^{j'}_{j}(f_{(i,\varphi)}(a_i))$. Together with (A3), this yields $f_{(i',\varphi')}(a_{i'}) \equiv emb^{j'}_{j}(f_{(i,\varphi)}(a_i))$. Since $\mathcal{N}_{\mathcal{J}}$ is direct, it follows that $f_{(i',\varphi')}(a_{i'}) \overset{p}{\mapsto} f_{(i,\varphi)}(a_i)$, as claimed. □

In a similar way as for Lemma A.8 we infer that if $\mathcal{M}_{\mathcal{I}}$ is direct and $\mathcal{N}_{\mathcal{J}}^{\mathcal{M}}$ inverse, then $\Pi(\mathcal{M}_{\mathcal{I}}, \mathcal{N}_{\mathcal{J}}^{\mathcal{M}})$ is inverse.

## A.2 Proof of Proposition 3.35

It remains to prove the last part of Proposition 3.35. Let $\mathcal{K}$ be $\Sigma_{i \in \mathcal{I}} \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$ in the subsequent lemmata.

**Lemma A.9** *If $(\mathcal{N}^A)_{A \in \mathcal{M}}$ is maximal over $(\mathcal{M}_{\mathcal{I}}, \mathcal{N}_{\mathcal{J}}^{\mathcal{M}})$, then $\Pi^{\mathfrak{D}}_{A \in \mathcal{M}} \mathcal{N}^A$ is maximal over $\Pi(\mathcal{M}_{\mathcal{I}}, \mathcal{N}_{\mathcal{J}}^{\mathcal{M}})$.*

*Proof* Let $F \in \Pi^{\mathfrak{D}}_{A \in \mathcal{M}} \mathcal{N}^A$ and $g_{(i,\varphi)} \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}$. We wish to apply Lemma 2.16 and assume that there are $\mathfrak{D}$-many stages $(i',\varphi') \in \mathcal{K}$ with $f_{(i',\varphi')} \overset{p}{\mapsto} g_{(i,\varphi)}$ for some $f_{(i',\varphi')}$ such that $F \overset{p}{\longmapsto} f_{(i',\varphi')}$. Let $\mathcal{H} \in \mathfrak{D}(\mathcal{K})$ denote this class of indices $(i',\varphi')$. We have to show $F \overset{p}{\longmapsto} g_{(i,\varphi)}$. By the definition of $\overset{p}{\longmapsto}$ it must be checked that

$$F(A) \overset{p}{\longmapsto} g_{(i,\varphi)}(a_i) \quad \text{whenever} \quad A \overset{p}{\longmapsto} a_i.$$

By (17) we have $d(\mathcal{H}) \in \mathfrak{D}(\mathcal{I})$. So there are $\mathfrak{D}$-many indices $i' \geq i$ such that $i' \in d(\mathcal{H})$ and $A \overset{p}{\longmapsto} a_{i'}$ for some $a_{i'} \in \mathcal{M}_{i'}$, since $\uparrow i \cap d(\mathcal{H}) \cap \mathcal{I}_A \in \mathfrak{D}(\mathcal{I})$. These elements satisfy $a_{i'} \overset{p}{\mapsto} a_i$ by (Factor), because $A \overset{p}{\longmapsto} a_{i'}$ and $A \overset{p}{\longmapsto} a_i$. For each such index $i'$, the fact that $i' \in d(\mathcal{H})$ shows $\mathcal{H}(i')(a_{i'}) \in \mathfrak{D}(\mathcal{J}^{a_{i'}})$ by Definition 3.28, and moreover, there is a function

$\varphi' \in \Pi^{\equiv}_{a_{i'} \in \mathcal{M}_{i'}} \mathcal{J}^{a_{i'}}$ with $(i', \varphi') \in \mathcal{H}$ by (12). So there are $\mathfrak{D}$-many indices $j' = \varphi'(a_{i'}) \in \mathcal{J}^{a_{i'}}$ with $f_{(i',\varphi')}(a_{i'}) \in \mathcal{N}_{j'}$ and

$$F(A) \overset{p}{\longmapsto} f_{(i',\varphi')}(a_{i'}) \overset{p}{\mapsto} g_{(i,\varphi)}(a_i).$$

This shows $F(A) \overset{p}{\longmapsto} g_{(i,\varphi)}(a_i)$ by Lemma 2.16 and the assumed maximality of $\mathcal{N}^A$ over $(\mathcal{N}^{a_i}_j)_{j \in \mathcal{J}^{a_i}}$. □

**Lemma A.10** *If $(\mathcal{N}^A)_{A \in \mathcal{M}}$ is extensional over $(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$, so is $\Pi^\mathfrak{D}_{A \in \mathcal{M}} \mathcal{N}^A$ over $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$.*

*Proof* We must prove for all $F, G \in \Pi^\mathfrak{D}_{A \in \mathcal{M}} \mathcal{N}^A$ that $F = G$, provided $Ext(F) \sim Ext(G)$. So let $A \in \mathcal{M}$ and we claim that $F(A) = G(A)$. We wish to appeal to extensionality of $\mathcal{N}^A$ over $(\mathcal{N}^{a_i}_j)_{j \in \mathcal{J}^{a_i}}$ to get $F(A) = G(A)$ by Lemma 2.19. Therefore, we have to show for some $a_i \in \mathcal{M}_i$ with $A \overset{p}{\longmapsto} a_i$, that $F(A) \overset{p}{\longmapsto} b_j$, $G(A) \overset{p}{\longmapsto} d_j$ implies $b_j \equiv d_j$ for all $j \in \mathcal{J}^{a_i}$. Let $\psi \in \Pi^{\equiv}_{a_i \in \mathcal{M}_i} \mathcal{J}^{a_i}$ be an arbitrary function with $\psi(a_i) = j$.

For all $(i, \varphi) \in (\mathcal{I} \times \uparrow \psi) \cap \mathcal{I}_F \cap \mathcal{I}_G \in \mathfrak{D}(\mathcal{K})$ we have $i \in d(\mathcal{I}_F \cap \mathcal{I}_G) \in \mathfrak{D}(\mathcal{I})$ by (17), and we can further assume that $i \in \mathcal{I}_A$, since both classes are in $\mathfrak{D}(\mathcal{I})$. An immediate consequence is $\varphi(a_i) \geq j$.

Let $f_{(i,\varphi)} \in Ext(F)$ and $g_{(i,\varphi)} \in Ext(G)$, then $f_{(i,\varphi)} \equiv g_{(i,\varphi)}$ follows by Lemma 2.19. Additionally, $i \in \mathcal{I}_A$ implies that $A \overset{p}{\longmapsto} a_i$ for some $a_i \in \mathcal{M}_i$, which moreover leads to $F(A) \overset{p}{\longmapsto} f_{(i,\varphi)}(a_i)$ and $f_{(i,\varphi)}(a_i) \overset{p}{\mapsto} b_j$. Similarly, we derive $g_{(i,\varphi)}(a_i) \overset{p}{\mapsto} d_j$. Finally, $f_{(i,\varphi)} \equiv g_{(i,\varphi)}$ allows us to infer $f_{(i,\varphi)}(a_i) \equiv g_{(i,\varphi)}(a_i)$, and hence $b_j \equiv d_j$. □

**Lemma A.11** *If $(\mathcal{N}^A)_{A \in \mathcal{M}}$ is complete over $(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$, so is $\Pi^\mathfrak{D}_{A \in \mathcal{M}} \mathcal{N}^A$ over $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$.*

*Proof* Let $\zeta$ be a consistent set of $\Pi(\mathcal{M}_\mathcal{I}, \mathcal{N}^\mathcal{M}_\mathcal{J})$. By the definition of a consistent set and (17) we know that $\mathcal{I}_\zeta \in \mathfrak{D}(\mathcal{K})$ and $d(\mathcal{I}_\zeta) \in \mathfrak{D}(\mathcal{I})$. We will define a limit element $F \in \Pi^\mathfrak{D}_{A \in \mathcal{M}} \mathcal{N}^A$ of $\zeta$ pointwise. Take $A \in \mathcal{M}$ and define for an element $a_i \in \mathcal{M}_i$ for which $A \overset{p}{\longmapsto} a_i$ and $i \in d(\mathcal{I}_\zeta)$ holds (there are $\mathfrak{D}$-many such elements due to $d(\mathcal{I}_\zeta) \cap \mathcal{I}_A \in \mathfrak{D}(\mathcal{I})$):

$$\begin{aligned} F(A) &:= \text{limit element of } \beta_{a_i} \\ \beta_{a_i} &:= \{g_{(i,\varphi)}(a_i) \in \mathcal{N}^{a_i}_{\varphi(a_i)} \mid \exists \varphi\, g_{(i,\varphi)} \in \zeta\}. \end{aligned} \tag{A4}$$

Completeness of $\mathcal{N}^A$ over $(\mathcal{N}^{a_i}_j)_{j \in \mathcal{J}^{a_i}}$ ensures that a limit element of $\beta_{a_i}$ exists. There are further properties that must be shown to guarantee that $F(A)$ is well-defined and that $F$ is indeed a limit element:

1. $\beta_{a_i}$ is a consistent set of the system $(\mathcal{N}^{a_i}_j)_{j \in \mathcal{J}^{a_i}}$,
2. $A \overset{p}{\longmapsto} a_i$ and $A \overset{p}{\longmapsto} a_{i'}$ implies $\beta_{a_i} \sim \beta_{a_{i'}}$ and
3. $Ext(F) \sim \zeta$.

1. First assume that $g_{(i,\varphi)}(a_i) \in \mathcal{N}^{a_i}_{\varphi(a_i)}$ and $g'_{(i,\varphi')}(a_i) \in \mathcal{N}^{a_i}_{\varphi'(a_i)}$ are both in $\beta_{a_i}$ and $\varphi(a_i) \leq \varphi'(a_i)$. Since $\zeta$ is a consistent set, $g'_{(i,\varphi')} \overset{p}{\mapsto} g_{(i,\varphi)}$ follows, and therefore $g'_{(i,\varphi')}(a_i) \overset{p}{\mapsto}$

$g_{(i,\varphi)}(a_i)$. Secondly, $\mathcal{I}_{\beta_{a_i}} \in \mathfrak{D}(\mathcal{J}^{a_i})$ must be shown. Since $i \in d(\mathcal{I}_\zeta)$ we have $\mathcal{I}_\zeta(i)(a_i) \in \mathfrak{D}(\mathcal{J}^{a_i})$, and moreover,

$$\mathcal{I}_{\beta_{a_i}} = \{\varphi(a_i) \in \mathcal{J}^{a_i} \mid \exists \varphi\, g_{(i,\varphi)} \in \zeta\} = \{\varphi(a_i) \in \mathcal{J}^{a_i} \mid \exists \varphi\, (i,\varphi) \in \mathcal{I}_\zeta\} = \mathcal{I}_\zeta(i)(a_i).$$

2. We show that the definition of $F(A)$ is independent of the chosen element $a_i$. By Lemma 2.19 it suffices to show $\beta_{a_i} \ni g_{(i,\varphi)}(a_i) \equiv g'_{(i',\varphi')}(a_{i'}) \in \beta_{a_{i'}}$ for $A \overset{p}{\longmapsto} a_i$, $A \overset{p}{\longmapsto} a_{i'}$ and $\varphi(a_i) = \varphi'(a_{i'})$. Consistency of $\zeta$ yields $g_{(i,\varphi)} \asymp g'_{(i',\varphi')}$, since both elements are in $\zeta$. Now

$$\uparrow(i,\varphi) \cap \uparrow(i',\varphi') \cap \mathcal{I}_A \cap \mathcal{I}_\zeta \in \mathfrak{D}(\mathcal{K}),$$

which implies that there exists an index $i'' \geq i, i'$, a function $\varphi'' \geq \varphi, \varphi'$ and elements $a_{i''} \in \mathcal{M}_{i''}$ and $g''_{(i'',\varphi'')} \in \zeta$ with $A \overset{p}{\longmapsto} a_{i''}$, $g''_{(i'',\varphi'')} \overset{p}{\mapsto} g_{(i,\varphi)}$ and $g''_{(i'',\varphi'')} \overset{p}{\mapsto} g'_{(i',\varphi')}$. Property (Factor) yields $a_{i''} \overset{p}{\mapsto} a_i$, hence $g''_{(i'',\varphi'')}(a_{i''}) \overset{p}{\mapsto} g_{(i,\varphi)}(a_i)$ and $g''_{(i'',\varphi'')}(a_{i''}) \overset{p}{\mapsto} g'_{(i',\varphi')}(a_{i'})$. Since $\varphi(a_i) = \varphi'(a_{i'})$ by assumption, we conclude that $g_{(i,\varphi)}(a_i) \equiv g'_{(i',\varphi')}(a_{i'})$.

3. We apply Lemma 2.19, so assume $F \overset{p}{\longmapsto} f_{(i,\varphi)}$ and $g_{(i,\varphi)} \in \zeta$ with $f_{(i,\varphi)}, g_{(i,\varphi)} \in \Pi_{a_i \in \mathcal{M}_i} \mathcal{N}^{a_i}_{\varphi(a_i)}$, and we have to show $f_{(i,\varphi)} \equiv g_{(i,\varphi)}$. Given $a_i \equiv b_i$, both in $\mathcal{M}_i$, so that it suffices to show $f_{(i,\varphi)}(a_i) \equiv g_{(i,\varphi)}(b_i)$.

Take $A := Emb_i(a_i) = Emb_i(b_i)$, then $A \overset{p}{\longmapsto} a_i$ and $A \overset{p}{\longmapsto} b_i$. First, $F(A) \overset{p}{\longmapsto} f_{(i,\varphi)}(a_i)$, second, $g_{(i,\varphi)} \in \zeta$ implies $(i,\varphi) \in \mathcal{I}_\zeta \in \mathfrak{D}(\mathcal{K})$, and hence $i \in d(\mathcal{I}_\zeta) \in \mathfrak{D}(\mathcal{I})$. By Definition (A4) we conclude that $F(A)$ is a limit element of $\beta_{b_i}$, i.e., $Ext(F(A)) \sim \beta_{b_i}$. Since $g_{(i,\varphi)}(b_i) \in \beta_{b_i}$ we infer that $f_{(i,\varphi)}(a_i) \equiv g_{(i,\varphi)}(b_i)$ by Lemma 2.19. This completes the proof. □

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