Formalizing Indefinite Extensibility

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Indefinite Extensibility

- A set is an extensionally definite totality of elements — definiteness is given by closure properties: union, Cartesian product, function space, . . .
- A class is a totality given as the extension of a predicate (informal notion). Every set is a class, but some classes are indefinitely extensible.

Two main examples

- Potential infinity: Definite means finite; indefinitely extensible means infinite. Only finite sets exist.

$$\text{Potential infinity} = \text{Indefinite extensibility} + \text{Finitistic view.}$$

- Infinitary view: Every totality not immediately leading to inconsistency is a set. Universes are proper classes.

Indefinitely Extensible Models

- Indefinite extensibility is often formalized logically, e.g. via modal or (semi-)intuitionistic logic. We use model theory only, independent of any inference rules.
- Ontological approach, not epistemological — in contrast to Kripke models.
- Set theory cannot serve as a foundation for these models: sets are extensionally fixed.
- We use extensible systems to formalize classes. These generalize both direct and inverse systems, with slightly different forms for FOL, STT, and DTT.

Indefinitely Extensible Models — Basic Ideas

- **Meta-level**: An idealized mathematician situated in time (context). Future states do not yet exist for her.
- Time is formalized as a directed index structure $(\mathcal{I}, \leq)$.
- **Object-level**: Objects and classes are dynamic.

$$\mathcal{M} \rightsquigarrow (\mathcal{M}_i)_{i \in \mathcal{I}},$$
$$a \rightsquigarrow (a_i)_{i \in \mathcal{I}'} \qquad \text{for indefinitely many } i \in \mathcal{I}' \subseteq \mathcal{I}.$$

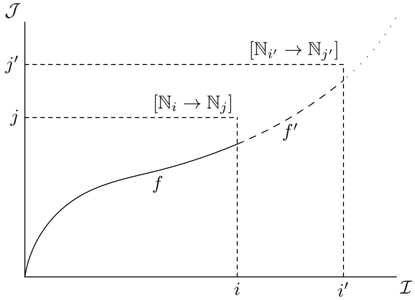
- Extensibility is more fundamental than any final completion.
- Completion is a context-dependent limit construction. Limits are sets inside the system, with different extensions in different contexts.
- Includes Dummett's intuition: quantifying over all objects of a kind creates a new object beyond them.

Indefinitely Extensible Models — Basic Ideas

- Extensible systems extend in two modes: compactified by a limit — then extensible and open again.
- Limits are given intensionally via a universal property, and extensionally via a concrete set construction.
- Subsumes coinductive definitions; similar in spirit to choice sequences, but without the epistemic component.
- Each (indefinitely extensible) type is interpreted as an extensible system.
- An extensible system is analogous to the cumulative set-theoretic universe — limits correspond to inaccessible ranks.

Example for the potential infinite

$[\mathbb{N}_i \to \mathbb{N}_j]_{i \to j \in \mathbb{N}^+ \times \mathbb{N}^+}$ is a system of finite function spaces where the class (function space) and its elements (functions) extend simultaneously.



The Formalization — Extensible Systems

- A predecessor relation $\overset{p}{\mapsto}$ serves as a non-symmetric equality across the different states of a changing object. The usual equality $\equiv$ is $\overset{p}{\mapsto}$ restricted to a single stage.
- Consistency: $a_i \asymp a_{i'}$ if there exists $a_{i''} \in \mathcal{M}_{i''}$ with $a_{i''} \overset{p}{\mapsto} a_i$ and $a_{i''} \overset{p}{\mapsto} a_{i'}$.
- A factorization property replaces transitivity:

$$a_{i'} \asymp a_i \iff a_{i'} \overset{p}{\mapsto} a_i \qquad \text{for all } i \leq i'.$$

  (Transitivity fails in general, e.g. for logical relations.)
- Indefinitely many is formalized as a filter $\mathfrak{D}(\mathcal{I})$ on an index class $\mathcal{I}$.

The Formalization — Further Structure

- Add embeddings (for FOL) or embedding-projection pairs (for STT and DTT).
- Limit construction: All properties and structure are extensible — a compactification by a limit inherits them.
- Concretely: a limit element is a maximal, consistent set of states.
- The indefinitely large relation $(i_0, \ldots, i_{n-1}) \ll i_n$ is needed to interpret the universal quantifier via an implicit reflection principle. The basic idea in FOL is the Löwenheim-Skolem construction.

The Indefinitely Extensible Model

Three components are required:

- Model: Extensible systems.
- Interpretation: Based on type theory.
- Logic: Principally the interpretation of quantifiers.

The interpretation has two parts — implicit (limit) and explicit (state):

An object a is represented as a pair (A, a_i), where $A \in \mathcal{M}$ is an implicit, context-dependent limit element and $a_i \in \mathcal{M}_i$ is an explicit state in the system.

Key idea

Every implicit reference to an object has an explicit “backup” at some stage.

State judgments

State judgments refine typing judgments. Example: state judgments for STT:

$$Typ^+ \ni \varrho^+ ::= \iota \mid \varrho^- \to \varrho^+ \quad \text{and} \quad Typ^- \ni \varrho^- ::= \delta \mid \varrho^+ \to \varrho^-,$$

with ι a base type and δ a definite base type. The type Ω of truth values has to be definite, e.g. $[\![\Omega]\!] = \{true, false\}$.
Let $C = (i_0, \dots, i_{n-1})$ and $D = (j_0, \dots, j_{m-1})$ be state contexts.

$$\frac{\mathsf{c} : D \to j \in \Sigma^{\mathcal{I}} \qquad C \vdash^\star \mathbf{r} : D}{C \vdash^\star \mathsf{c}\mathbf{r} : j} \qquad \frac{j \geq i_k : \varrho^+}{C \vdash^\star \mathsf{x}_\mathsf{k} : j}$$

$$\frac{j \leq i_k : \varrho^-}{C \vdash^\star \mathsf{x}_\mathsf{k} : j} \qquad \frac{C.i \vdash^\star \mathsf{r} : \Omega \qquad C \ll i}{C \vdash^\star \forall \mathsf{x}^\varrho \mathsf{r} : \Omega}$$

$$\frac{C \vdash^\star \mathsf{r} : i \to j \qquad C \vdash^\star \mathsf{s} : i}{C \vdash^\star \mathsf{rs} : j} \qquad \frac{C.i \vdash^\star \mathsf{r} : j}{C \vdash^\star \lambda \mathsf{x}^\varrho_\mathsf{n} \mathsf{r} : i \to j}$$

The Type-Theoretic Language

- Only terms admitting a state judgment are interpretable. A term may lack a state judgment if there are not indefinitely many stages at which it can be interpreted.
- In general, variables of function type $\varrho \to \sigma$ (both types indefinitely extensible) have no interpretation — a strong ontological assumption.
- However, a relational description $\varrho \to \sigma \to \Omega$ together with a proof of functionality is possible.
- The finitistic view imposes more restrictions, since more types are indefinite extensible.
- We allow *type* : *type*. Inconsistencies are avoided by disallowing variables of function type between indefinitely extensible types (e.g. Hurkens' powerful universe requires them).

Status

	Model	Interpretation	Logic
FOL	RML 2022	RML 2022	RML 2022
STT	TCS 2023	LIPIcs 2024	unpublished
DTT	to appear	in progress	open

Table: Current status.

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Any questions?

Thank you for your attention.